# Fuzzy Approaches to Proportional Fairness

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***Abstract*—Proportional fairness has been shown to maximize the aggregate utility of rate control for elastic traffic in a resource sharing communication network, and has been applied to a broad range of resource allocation problems. For a refined analysis, however, the representation of proportional fairness as a relation between vectors with positive components will often not provide the level of detail that is needed. Therefore, we study the representation as a fuzzy relation, and propose several ways to specify a measure function to allocate a degree of (proportional) relatedness. The approaches are based on combinatorial aspects, especially size and number of related subvectors, and geometric aspects, especially the minimum effort to change a vector to become related. A case study demonstrates that the introduced fuzzy fairness relations can be used to numerically evaluate the suitability of the maxmin fair state to represent the proportional fair state in a resource sharing network problem with maximum link capacities.**

***Keywords*-fairness; proportional fairness; maxmin fairness; fuzzy relation**

## I. Introduction

Fuzzy relations have been studied within the framework of fuzzy sets for a long time. The formal proposal was given by Zadeh in his seminal paper [1], but today, each fundamental textbook on fuzzy theory will contain a special chapter on fuzzy relations (for example [2]). The goal of introducing *fuzzy relations* is to numerically evaluate linguistic statements like "A is much larger than B" or "A is a little bit larger than B" in a rigid and consistent manner. Within a pure set theoretic context (where it is often called a *crisp* relation), such terms would equally appear as "A is larger than B" and a further refinement is not possible. More formally, a fuzzy relation is defined as a mapping $\mu : X \times Y \to [0, 1]$, where $X$ and $Y$ are arbitrary sets, $X \times Y$ is the direct product of these sets, i.e. the set of all pairs $(x, y)$ with $x \in X$ and $y \in Y$. The *membership function* $\mu$ assigns to each pair $(x, y)$ from $X \times Y$ a membership value $\mu[(x, y)]$, i.e. a real number from the interval $[0, 1]$. A membership value 1 is the counterpart of the classic crisp relation between $x$ and $y$. Remember that a (crisp) relation is defined as a subset $R$ of $X \times Y$, where $(x, y) \in R$ means that $x$ and $y$ are in relation $R$ to each other, or $xRy$. Other possible notations are $x >_R y$ if we focus on ordering, $x \sim_R y$ if we focus on similarity, or $x =_R y$ if we focus on equality. For example, "A is much larger than B" could be indicated by $\mu_{larger}[(A, B)] = 0.9$ and "A is a little bit larger than B" by $\mu_{larger}[(A, B)] = 0.02$. If the sets $X$ and $Y$ are the same, we also speak about a *binary* fuzzy relation.

Fuzzy relations found numerous applications in many fields. However, under given circumstances, the question is often how to specify the mapping $\mu$. From its definition alone, there are no further requirements, and any possible mapping into $[0, 1]$ is feasible. Then, practical aspects have to be weighted, as well as the integration of a fuzzy relation in a superposed evaluation framework.

Proportional fairness has been shown to maximize the aggregate utility of rate control for elastic traffic in a resource sharing communication network [3] and has been applied to a broad range of resource allocation problems. The proportional fair state is characterized as a state vector $x$ of $n$ positive-valued traffic rates such that for any other feasible state $y \in R_n^+$ the inequality

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} \leq 0 \tag{1}$$

holds. This gives rise to the definition of a *proportional fair dominance relation* between two $n$-dimensional vectors $x$ and $y$ with positive components. It is said that $x$ proportional fair dominates $y$ ($x >_{pf} y$) if and only if the *indicator expression* $\sum_i (y_i - x_i)/x_i$ is not positive, or alternatively, if $\sum_i y_i/x_i \leq n$.

In this work, we want to introduce and study various ways of providing the fuzzy counterpart of this relation. We can consider various network resource allocation problems, where the availability of a fuzzy version of proportional fairness is of advantage. Later on in the paper, we will exemplify how fuzzy proportional fairness allows for numerically establishing claims like "the maxmin fair state is nearly as much proportional fair as the proportional fair state" for a specific routing in a resource sharing communication network. For fuzzification of a relation, tractability has to be taken into account as well. Often, a simple formulation of a "measure" for fulfilling a relation can guide to intractable problems, either due to combinatorial explosion of the number of choices, or appearance of analytic expressions that cannot be solved in a closed form. It is also of importance to paraphrase a fuzzy relation in such a way that the computation of the membership degree is tractable. This will be the topic of section II of the paper, where we are introducing a combinatorial approach, and alternatively a geometric approach to fuzzy proportional fairness. The example study for using fuzzy proportional fairness in a resource sharing communication network will be provided in section III. The paper ends with a short conclusion.

## II. Fuzzy proportional fairness

In the following, we are considering fuzzy measures for a numerical representation of the degree, by which a positive vector $x \in R_n^+$ proportional fair dominates another positive vector $y$. For convenience, we will write $\mu(x, y)$ instead of $\mu[(x, y)]$ while keeping in mind that the order of arguments is relevant. Then, $\mu(y, x)$ describes the opposite situation, the degree by which $y$ dominates $x$.

### A. Combinatorial approaches

In a combinatorial approach, we consider corresponding subsets of the components of $x$ and $y$. The measures are based on counts on these subsets. More formally, we define corresponding *subvectors*.

Given are two vectors $x$ and $y$, each with $n$ positive components, and a subset $I \subseteq \{1, \ldots, n\}$ of indizes (index set). Obviously, there are $2^n$ such index sets. Then, $x_I$ denotes the subvector of $x$, which is composed from components $x_i$ of $x$ with $i \in I$. For example, if $x = (2, 3, 5, 8)$ and $I = \{1, 3\}$ then $x_I$ is the vector containing only the first and third component of $x$, i.e $x_I = (2, 5)$. The size of a subvector is the number of its components, or $|I|$.

*1) Definitions:* We introduce three ways of counting *subvector dominance* (SVD), which will be called total, internal, and external SVD. For evaluating *total SVD* (written as $\mu_{t-SVD}$), we take the size of the *largest* subvector of $x$ proportional fair dominating the corresponding subvector of $y$ and divide by $n$. Thus, if $x >_{pf} y$ then $\mu_{t-SVD}(x, y) = 1$ since there is a subvector of $x$ of size $n$ that dominates $y$: $x$ itself.

For the internal and external SVD, we consider proportional fair dominance between the subvectors themself. For the internal SVD, we count the number of index sets $I$ for which $\sum_{i \in I} y_i/x_i \leq |I|$, i.e. where the subvectors of $x$ proportional fair dominate the corresponding subvectors of $y$ according to their length. For also taking the "true" dimension of the problem into account, the external SVD counts the number of index sets, for which $\sum_{i \in I} y_i/x_i \leq n$. In both cases, we divide the count by $2^n$, the total number of index sets, in order to yield a value between 0 and 1.

*2) Tractability and suitability of SVD:* The computation of the total SVD is rather simple. We denote the values $y_i/x_i$ as $a_i$ and the set of all $a_i$ as $a$. Assume the ratios of $a_i$ to be sorted in non-decreasing order, and the subscript $a_{(i)}$ indicates the $i$-th smallest element of the $a_i$ according to this sorting. Then, for the existence of a subvector with $k$ elements that proportional fair dominates the corresponding subvector of $y$, it is necessary and sufficient that the sum of the first $k$ $a_{(i)}$-ratios is smaller or equal to $k$. It is sufficient since we can use the subvector for the index set $I = \{(1), (2), \ldots, (n)\}$ which fulfills the condition. It is also necessary, since the sum of any $k$ $y_i/x_i$-ratios is larger or equal to the sum of the $k$ smallest ratios, and from the former sum being smaller or equal to $k$ it also follows that $\sum_{i=1}^{k} a_{(i)} \leq k$.

Thus, for computing the total SVD, we just have to find the largest $k$ with this property:

$$\mu_{t-SVD}(x, y) = \frac{1}{n} \max_{k=1,\ldots,n} \{k \mid \sum_{i=1}^{k} y_{(i)}/x_{(i)} \leq k\} \qquad (2)$$

If no $k$ fulfills the condition, the measure is 0. The computation of total SVD is tractable, the computational effort grows linearly with $n$.

The total SVD can take values between 0 and 1. If all $y_i > x_i$ then there is even no single-element subvector of $x$ dominating the corresponding $y$-subvector, and the measure is 0. If $x >_{pf} y$ then the largest $k$ equals $n$ and the measure is 1. However, the main disadvantage of this measure is the small number of levels. For $n$-dimensional vectors, the measure will only have $(n + 1)$ different values.

The computation of the external SVD measure appears to be a hard counting problem (#P-complexity) [4], [5]. The task is to count all possible subsets of $a$ such that the sum of its elements is smaller or equal to $n$. For smaller $n$ an exhaustive search could be made over all possible $2^n$ subvectors to compute this number. However, for larger $n$ it is better to consider an approximative method.

One approach is based on the so-called Markov Chain Monte Carlo method [4]. First we define a Markov process on a state vector $s$, which is initialized with $s = \{0, \ldots, 0\}$ with $n$ components, and some number $b$. Then, this vector repeatedly changes into a new state according to the following rules:

1) With probability 0.5 do not change state.
2) Otherwise, select a random element $s_j$ of $s$ and change it to $1 - s_j$ to get another state $s^*$.
3) If for the resulting state $\sum_{i=1}^{n} s_i^* a_i \leq b$ then set $s = s^*$, otherwise keep former $s$.

The state vector will describe an index set (selecting all indices $i$ where $s_i = 1$). To use this as an estimate for the number of subsets is based on the following approach. First, the $a_i$ are sorted in increasing order, such that $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$. Then, $(n + 1)$ values $b_i$ are defined as follows:

- $b_0 = 0$
- $b_i = \min\{n, \sum_{j=1}^{i} a_{(j)}\}$ for $1 \leq i \leq n$.

For some $b$, $N_b$ denotes the number of subsets of $a$ with element sum smaller or equal to $b$ (omitting $a$ in the notion for simplicity). Then one can rewrite

$$N_n = \frac{N_{b_n}}{N_{b_{n-1}}} \times \frac{N_{b_{n-1}}}{N_{b_{n-2}}} \times \cdots \times \frac{N_{b_1}}{N_{b_0}} \times N_{b_0} \qquad (3)$$

At first note that $N_{b_0} = 1$ (the only choice is to set all $s_i$ to 0). But the other factors could be sampled by above Markov process: to estimate $N_{b_i}/N_{b_{i-1}}$ one uses $b_i$ as $b$-value in the process, and after $r$ repetitions of the state change, it is checked if the final state is also in the set of subsets with element sum less or equal to $b_{i-1}$ (note that this is a smaller value).

This method was tested on random cases for values of $n$ from $\{5, 10, 15, 20\}$. The number of iterations of the Markov process has been set to 100. For each case of $n$, 30 pairs of vectors $x$ and $y$ were generated, and the estimate according to Eq. (3), where each factor was sampled 100 times, was computed 10 times and averaged. The ratio of the exact value to the average estimate then was averaged over the 30 runs. The results are shown in Table I.

Table I
RESULTS FOR APPROXIMATING THE NUMBER OF SUBSETS.

| dimension | average ratio | sdev |
|---|---|---|
| 5 | 0.95 | 0.05 |
| 10 | 1.0 | 0.13 |
| 15 | 1.07 | 0.14 |
| 20 | 1.38 | 0.2 |

It can be seen that the method in this setting is becoming less acceptable for $n = 20$, while for lower $n$, the exact solution might be evaluated much faster. However, the value of $r$ has a strong influence. If we increase $r$ to 500 for the case $n = 20$, the computation time is still much below the time needed for the exact computation, but the average ratio becomes $0.98 \pm 0.16$. The number of samples for the single factors has lower influence. For example, changing to 1000 for $n = 20$ gives a similar result ($1.38 \pm 0.18$). Table II lists a few randomly picked up results for the case of 20 dimensions, in order to provide a visual impression of the quality of the approximation.

Table II
SOME NUMERICAL EXAMPLES FOR APPROXIMATING THE NUMBER OF SUBSETS.

| Exact value | Estimated value |
|---|---|
| 247561 | 224006 |
| 812056 | 805207 |
| 437689 | 465182 |
| 517380 | 381953 |
| 260550 | 226550 |
| 256368 | 303769 |
| 539113 | 436566 |
| 134078 | 104157 |
| 524016 | 509931 |
| 1048576 | 888308 |
| 1042919 | 884490 |

This setting ($r = 500$ state changes of Markov chain, 100 samples per factor in Eq. (3), averaging over 10 estimates) seems to be reasonable in case of 20 or more dimensions, including consideration of a corresponding increase of $r$ for larger $n$.

Thus, the external SVD is not tractable, but can be approximated. If for all $i$ $y_i > nx_i$, then there is no subvector of $x$ proportionally fair dominating the corresponding subvector of $y$ and the measure becomes 0. If on the other hand all $x_i > y_i$, then each subvector dominates, and the measure becomes 1. However, $x >_{pf} y$ alone is not sufficient to have a measure of 1.

The internal SVD is similar to the external SVD. The number of subvectors to test grows exponentially, but an approximation similar to the one given for the external SVD might be devised. Also here, values can range from 0 to 1, while the measure is not necessarily 1 for the case of proportional fairness. The external SVD will usually be much larger than the internal SVD, due to the relaxed condition for the indicator expression.

### B. $\epsilon$-Dominance measure: a geometrical approach

In a situation where a vector $x$ does not proportional fair dominate $y$ we might consider the least effort to either modify $x$ to $x^*$ or $y$ to $y^*$, or both, in order to achieve $x^* >_{pf} y^*$. The relation of the needed strength of change to the magnitude of the vectors themself then gives another fuzzy measure. For reasons of analytic tractability, we consider here only a change of $y$ and keep $x$ fixed. We introduce the fuzzy measure $\mu_\epsilon(x, y)$ as follows. If $x >_{pf} y$ then $\mu_\epsilon(x, y) = 1$. Otherwise, each component $y_i$ of $y$ is modified by subtracting $\delta_i$ (note that in theory, $\delta_i < 0$ is possible). The choice of $\delta_i$ has to fulfill:

$$\sum_{i=1}^{n} \frac{y_i - \delta_i}{x_i} = n \tag{4}$$

and the goal can be formulated as:

$$\text{Minimize } \sum_{i=1}^{n} \delta_i^2, \text{ s.t. Eq. (4)} \tag{5}$$

Using the method of LAGRANGE multipliers, we introduce the function

$$F(\delta_i; \lambda) = \sum_{i=1}^{n} \delta_i^2 + \lambda \left( \sum_{i=1}^{n} \frac{y_i}{x_i} - \sum_{i=1}^{n} \frac{\delta_i}{x_i} - n \right) \tag{6}$$

and we have to solve the set of equations

$$\begin{aligned} \frac{\partial F}{\partial \delta_i} &= 2\delta_i - \frac{\lambda}{x_i} = 0 \\ \delta_i &= \frac{\lambda}{2x_i} \end{aligned}$$

Introducing these $\delta_i$ in the condition that is given by Eq. (4) gives $\lambda$:

$$\sum_{i=1}^{n} \frac{y_i}{x_i} - \sum_{i=1}^{n} \frac{\lambda}{2x_i^2} = n$$
$$\lambda = \frac{\sum_i y_i/x_i - n}{\sum_i 1/2x_i^2}$$

and so we achieve

$$\delta_k = \frac{\sum_{i=1}^{n}(y_i/x_i) - n}{x_k \sum_{i=1}^{n}(1/x_i^2)} \tag{7}$$

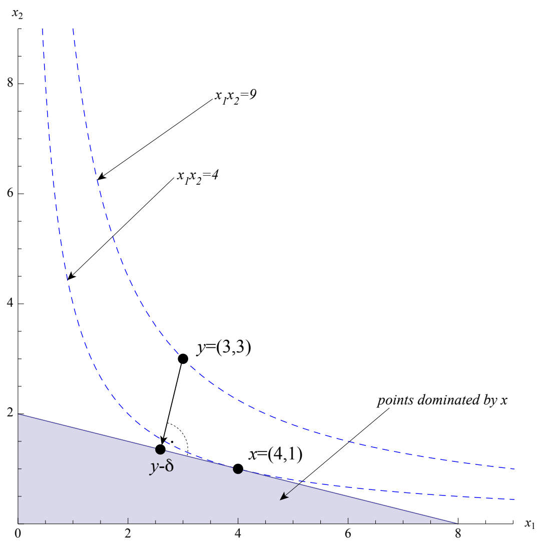


Figure 1. Example for $eps$-dominance measure. The point $y = (3, 3)$ needs to be shifted by $\delta = (-28/4 \cdot 17, -28/1 \cdot 17)$ in order to become proportional fair dominated by $x = (4, 1)$. The shifted point corresponds to the perpendicular of the point $y$ onto the line $(8-x)/4$ that bounds the set of points proportional fair dominated by $x$. The measure is $1 - |\delta|/|y| \approx 0.6$.

Figure 1 shows an example for the case of 2 dimensions. The point $x = (4, 1)$ does not proportional fair dominate $y = (3, 3)$. However, after shifting $y$ to $y - \delta$, $x >_{pf} y - \delta$ holds. The shortest $\delta$ with this property corresponds to the length of the perpendicular onto the line that bounds the points proportional fair dominated by $x$. In this case, it is computed to $\delta = (28/4 \cdot 17, 28/17)$. We define the ratio of the length of the shortest $\delta$ to the length of $y$, subtracted from 1, as the $\epsilon$-dominance measure.

$$\mu_\epsilon(x, y) = \begin{cases} 1 & \text{if } x >_{pf} y \\ 1 - \sqrt{\frac{\sum_{k=1}^{n} \delta_k^2}{\sum_{i=1}^{n} y_i^2}} & \text{otherwise} \end{cases} \tag{8}$$

where $\delta_k$ is given by Eq. (7).

## III. Case study

We demonstrate the use of fuzzy proportional fairness for fair traffic allocation in a resource sharing communication network. The network is represented as a graph, and end-to-end traffic between nodes of the graph has to be routed via links through the network. In this model, it is assumed that links between nodes have maximum capacities. The routing of traffic along paths in the network can result in shared links, thus causing congestion if traffic becomes too high. For more details of this model, see e.g. [6].

A traffic allocation is called *maxmin fair*, if for every increase of one traffic rate there is another traffic rate, of same amount or smaller, that has to be decreased in order to avoid congestion at some link. It is known that for this model there is exactly one maxmin fair allocation, and it can be found by the Bottleneck Flow Control (BFC) algorithm [7]. On the other hand, in such a resource sharing network there is also the proportional fair state, which is also unique and characterized by maximizing total logarithmic utility of traffic rates.

Often, both fair states do not coincide, especially if there is much link sharing in the network due to high load of end-to-end user traffic. However, for the maxmin fair state, we can use the rather simple BFC algorithm, which has linear complexity and serves exact solutions. If we want to find the proportional fair state, we usually need to solve a non-linear optimization problem with linear inequality constraints and are obtaining only numerical approximations to the solutions. The question then is if the higher effort to find the proportional fair state is really needed, and if the maxmin fair state cannot serve as “proxy” of proportional fairness as well.

Using fuzzy proportional fairness, we can study the validity of such claims. Consider a fully connected graph with 6 nodes, and two routings with 10 paths: routing 1 is given by the 10 node sequences

(4 6 5 2), (3 4 6 5), (1 2 3), (4 6 5 2 3), (3 4 6),
(2 3), (5 2 6 1), (5 2), (3 4 6 5 2), (5 4)

noindent and routing 2 is given by

(2 4 5), (2 4), (2 6), (1 5), (3 4),
(5 4 6), (5 4 6 2), (4 6), (3 1 2), (4 3).

Routing 1 has longer paths and thus higher link sharing then routing 2. All maximum capacities in the network have been set to 100 (rate units).

The maxmin fair states for routing 1 and 2 can be found by the BFC algorithm as $mmf_1 = (20, 20, 40, 20, 20, 40, 20, 20, 20, 100)$ and $mmf_2 = (33, 66, 66, 100, 50, 33, 33, 33, 100, 50)$. The proportional fair states can be found by using a numerical optimizer[1]: $pf_1 = (15, 30, 44, 11, 30, 44, 30, 30, 15, 100)$ and $pf_2 = (43, 57, 76, 100, 50, 34, 24, 43, 100, 50)$. The values here have been rounded. It can be seen, as it should be, that $pf_{1,2} >_{pf} mmf_{1,2}$ and $mmf_{1,2} >_{mmf} pf_{1,2}$ ($>_{mmf}$ is the maxmin fair dominance relation). This is all information that we can get if using crisp relations alone.

[1] Here, we used the NOptimize[]-function of Wolfram *Mathematica*®8.

Table III
FUZZY FAIRNESS VALUES FOR THE EXAMPLE PROBLEMS, WHERE $mmf$ IS THE MAXMIN FAIR STATE, AND $pf$ IS THE PROPORTIONAL FAIR STATE.

| Measure | Routing 1 | Routing 2 |
|---|---|---|
| $\mu_{t-SVD}(mmf, pf)$ | 0.7 | 0.8 |
| $\mu_{i-SVD}(mmf, pf)$ | 0.158203 | 0.234375 |
| $\mu_{i-SVD}(pf, mmf)$ | 0.529297 | 0.578125 |
| $\mu_{e-SVD}(mmf, pf)$ | 0.993164 | 0.999023 |
| $\mu_{\epsilon}(pf, mmf)$ | 1.0 | 1.0 |
| $\mu_{\epsilon}(mmf, pf)$ | 0.93222 | 0.973381 |
| $\mu_{svd}^{mmf}(pf, mmf)$ | 0.125 | 0.46875 |
| $\mu_{svd}^{mmf}(mmf, pf)$ | 0.876953125 | 0.546875 |

Table III list some fuzzy fairness measures. It also includes the fuzzy maxmin fairness relation that was introduced in [8], and which corresponds to the internal SVD. We can see that the internal SVD for both, proportional fairness and maxmin fairness, has rather low values, but the values are lower for the proportional fair state pretending a maxmin fair state than for the maxmin fair state pretending a proportional fair state in case of routing 1 (higher link sharing) and larger for routing 2. Moreover, the geometric measures are very close to 1.

To consider this point in more detail, we also sampled probability distributions of the measures for routing 1. Figure 2 shows the distribution of the fairness degree measures including random vectors. For both plots, 100,000 feasible random vectors have been sampled, and the frequencies of fuzzy fairness measures have been sorted into bins of size 0.01. It can be seen that the $\epsilon$-dominance fuzzy proportional fairness for pairs of random vectors follows a normal distribution. The increase at the end for the measure 1 corresponds to the fact that the degree is set to 1 if one sample vector proportional fair dominates the other, and there is no need to shift the second vector anymore. In case that one vector is the proportional fair state, the fuzzy measure for the second vector for proportional fair dominating $pf_1$ is clearly becoming smaller, the standard deviation is nearly same, and the sharp peak for measure 1 vanishes.

We consider the same situation, but now sample the distribution of the $eps$-dominance measure between the maxmin fair state $mmf_1$ and a random vector. Since the numerical values do not fit with the scale of Fig. 2, the only non-zero values are listed in Table IV.

Note that the same evaluation for proportional fairness would always give the value 1. The high similarity of the distribution for proportional and maxmin fairness is notable. In more than 90% of the cases, the maxmin fair state proportional fair dominates feasible traffic allocations, and then the distributions falls very sharply for even small differences to 1.

We can conclude and provide a number of arguments that maxmin fairness can be used as proportional fairness "proxy," if the focus is on the geometric distance to the

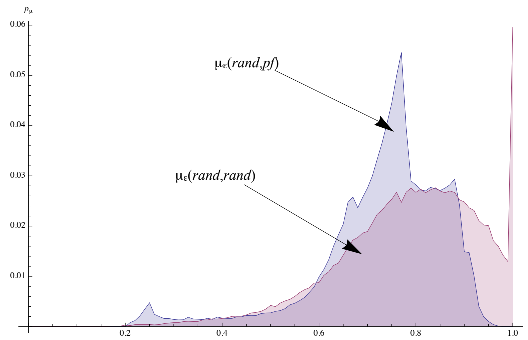


Figure 2. Distribution of $eps$-dominance measures between two random traffics for the example problem (red curve) and between a random point and the proportional fair state (blue curve).

Table IV
DISTRIBUTION OF THE $eps$-DOMINANCE MEASURE BETWEEN THE MAXMIN FAIR STATE AND A RANDOM TRAFFIC FOR THE EXAMPLE PROBLEM.

| Measure | Probability |
|---|---|
| $0.0 \sim 0.86$ | 0.0 |
| 0.87 | 0.0001 |
| 0.88 | 0.0003 |
| 0.89 | 0.0007 |
| 0.90 | 0.00125 |
| 0.91 | 0.00174 |
| 0.92 | 0.00301 |
| 0.93 | 0.00375 |
| 0.94 | 0.00634 |
| 0.95 | 0.00908 |
| 0.96 | 0.01104 |
| 0.97 | 0.01429 |
| 0.98 | 0.01839 |
| 0.99 | 0.02039 |
| 1.0 | 0.90944 |

proportional fair state, in comparison to distances to other feasible traffic allocations. From a grouping point of view (expressed by subvector dominance), the choice is also possible, but less favourable.

## IV. CONCLUSION

In order to provide a fuzzy relation for proportional fairness, we have to specify a measure for the degree of one vector proportional fair dominating another vector. We have proposed several ways to accomplish this goal, based on combinatorial and geometric aspects. Combinatorial approaches refer to the number and size of subvectors dominating corresponding subvectors of the other vector. Geometric approaches refer to the spatial location of the two vectors. Here, we focused on the minimal effort to modify one vector in order to become proportional fair dominated. The tractability and suitability of such measures has been discussed, and it was shown that either all proposed measures are computationally tractable, or at least a suitable approximation procedure is available. To demonstrate the use of such fuzzy fairness relations, we have compared the maxmin fair state and proportional fair state in a resource sharing communication network. For two examples it could be evaluated that the maxmin fair state (which can be easily found by the Bottleneck Flow Control algorithm) can serve as fairly good representation of the proportional fair state as well (which can only be numerically approximated).

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