# Soft Data Fusion by N-th Degree T-Norms

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## Abstract

*The extension of the set operation mink (maxk) of selection of the k-th smallest (largest) element to fuzzy sets is considered. The extension is based on the provision of set operators purely based on subset selection, and maximum and minimum operators. Then, all or part of all maximum and minimum operators can be replaced by corresponding pairs of T- and S-norms. There are only a few cases where such an approach is still preserving sufficient mathematical properties, including recursiveness of the mink operation. The approach based on* $min2 = max(min(a_j|j \neq i)|1 \leq i \leq n)$ *for a set* $(a_i)$ *of n real values from* $[0,1]$ *is considered in more detail, and applied in an example manner to the definition of an image operator. The fact that Tk-norms provide larger values than the (often very small values of) T-norms, from which they are derived, offers a lot of practical applications for Tk-norms, e.g. in fuzzy control, neuro-fuzzy systems, and computing-with-words.*

## 1 Introduction

Triangular norms, or T-norms for short, and their corresponding S-norms (or T-conorms) play an important role in the formal specification of fuzzy concepts, be it to compute the section of fuzzy membership functions, the logical *And* operation in computing-with-words, the output of neuro-fuzzy networks [4], the defuzzification value in fuzzy control [1], the specification of fuzzy morphology operations in image processing or the fuzzy fusion of data [3] [6]. While the totality of all functions comprising T-norms is mathematically yet unknown, several basis definitions and representation theorems are already known (see [2] for a comprehensive study on triangular norms). However, a major drawback in the application of T-norms are their low numerical values, when they are computed from a larger number of data. To give an easy example: the algebraic (or product norm) of $n$ data with values 0.1 is $10^{-n}$, while the standard T-norm minimum still computes to 0.1. T-norms are related to the mimimum operation in many ways, as well as S-norms are related to the maximum operation. To overcome this problem, and following [5], we propose to consider counterparts to the $n$-th smallest (or $n$-th largest) element of a set of data in a similar manner as T-norms are considered counterparts of the mimimum operation (e.g. by preserving relations like $T(a,1) = a$). It comes out that there are several ways to provide appropriate definitions of Tk-norms, but only a few of them preserve a sufficient number of mathematical properties that make them applicable in a similar fashion like T- and S-norms. Such definitions are discussed in section 2 of this paper, while section 3 solicits a particular definition and gives some of their properties. The advantage of providing higher numerical values is demonstrated in section 4, when image processing operators are based on the Tk-norm.

## 2 Algebraic Approaches to Set Ranking

In this section, we are considering approaches to set ranking functions based on non-fuzzy concepts, and are discussing their possible fuzzification. Given a finite set $A$ of real numbers, the set function $\min(A)$ assigns the minimum value of all elements of $A$ to $A$ itself. In order to get the second smallest value of $A$, denoted by $min2(A)$, it seems to be sufficient to consider an expression like

$$min2(A) = \min(A \setminus \{\min(A)\}). \tag{1}$$

The problem with the defuzzification here is that we do not have a unique means for fuzzifying the set $\min(A)$ as well as the set difference $A \setminus B$, once $A$ and $B$ are fuzzy sets. It will later come out that the approach presented in this paper actually gives such an extension, but the starting point is different.

In any case, the *recursive* nature of such ranking operators has to be taken into account. So, considering the *third smallest element* of a set $A$, it can be defined as the second smallest element of a set without the smallest or second smallest, or as smallest of a set without the smallest *and* second smallest element as well.

In [5] a different expression is considered to obtain the so-called *fuzzy OR-2* operator. The expression serving the second largest element $max2(A)$ of a set $A = (a_1, a_2, \ldots, a_n)$ of $n$ real values is purely based on using subsets, maximum and minimum operators:

$$max2(A) = \max_{1 \le i \le n} [\min(a_i, \max_{1 \le j \le n, j \ne i} (a_j))] \tag{2}$$

where a corresponding definition for the second smallest element can be provided as well:

$$min2(A) = \min_{1 \le i \le n} [\max(a_i, \min_{1 \le j \le n, j \ne i} (a_j))] \tag{3}$$

Then, in [5] it is proposed to replace any occurrence of the minimum and maximum operator by a T-norm and its corresponding S-norm (in particular, the HAMACHER T-norm is considered, and the application is to base feature selection on it).

The approach offers two major drawbacks: one is that the resulting expression is rather complex, and it becomes very hard to establish any mathematical properties of such expressions. The other drawback is the potential extension to the following ranking cases, i.e. when establishing corresponding definitions for the third, fourth a.s.f. smallest element of a set. In the given approach, the recursive property of set ranking is not preserved.

To see this in more detail, consider the corresponding extension of eq. 3 to the case of computing the $k$-th smallest element $mink(A)$ of $A$ (with $1 \le k \le n$) of real numbers from $[0,1]^n$. With

$$I_l = \{i_j \,|\, 1 \le j \le (k-1), \\ 1 \le i_1 < \ldots < i_{k-1} \le n\}$$

and the index $l$ running over all choices of $(k-1)$ elements from the set $(1, 2, \ldots, n)$ with values from 1 to $\binom{n}{k-1}$, and:

$$S_l = \{a_i \,|\, i \in I_l\} \tag{4}$$

we can define

$$mink(A) = \min_{I_l} (\max [\max(S_l), \min_{m \notin I_l} (a_m)]) \tag{5}$$

This (rather complex) expression yields the $k$-th smallest element of the set $A$. However, the expression in eq. 3 can be recursively extended as well:

For $k = 1$:

$$min1(A) = \min(a_1, \ldots, a_n) \tag{6}$$

For $1 < k \le n$:

$$mink(A) = \min_{i=1,\ldots,n} (\max [a_i, \\ min(k-1)\{a_j \,|\, 1 \le j \le n, j \ne i\}]) \tag{7}$$

and $mink(A) = 1$ for $k > n$.

The point is that both definitons do not match when the minimum and maximum operators are replaced by corresponding T- and S-norms. With a little effort, this can be seen when using e.g. the *product norm* as T- and S-norms, but the evaluation will be omitted here.

In this paper, we propose the usage of a simpler version of eq. 3 and the formal replacement of T-norms in a manner preserving recursiveness. Instead of eq. 3 we consider the expression

$$min2(A) = \max_i \left( \min_{j \neq i}(a_j) \right) \tag{8}$$

with $1 \leq i, j \leq n$. This (simpler) expression also yields the second smallest element of the set $A$, and it can be recursively extended to the case of the $k$-th smallest element by

$$min1(A) = \min(a_1, a_2, \ldots, a_n) \tag{9}$$

for $k = 1$,

$$min(k+1)(A) = \max_i \left( mink\{a_j \,|\, j \neq i\} \right) \tag{10}$$

with $1 \leq i, j \leq n$ for $1 \leq k < n$ and, as before, $mink(A) = 1$ for $k > n$.

We may also consider a non-recursive extension. With an *index set* of size $k - 1$

$$I_l = \{i_j \,|\, 1 \leq j \leq (k-1), \\ 1 \leq i_1 < \ldots < i_{k-1} \leq n\}$$

and the index $l$ running over all choices of $(k-1)$ elements from the set $(1, 2, \ldots, n)$ with values from 1 to $\binom{n}{k-1}$ and:

$$S_l = \{a_i \,|\, i \in I_l\} \tag{11}$$

we can define

$$mink(A) = \max_{I_l} \left( \min_{m \notin I_l}(a_m) \right) \tag{12}$$

If we were going to use a pair of corresponding T- and S-norms for both, the maximum and minimum operator, the definitions of eqns. 10 and 12 would not match. To see this, consider four values $a, b, c$ and $d$ from $[0, 1]$ and the extended recursive definition of eq. 12, with Tk denoting the extended definition, for the case $k = 3$ $(1 \leq i, j, k \leq n)$:

$$\begin{aligned} T3(a,b,c,d) &= S_{i,j}(T\{a_k \,|\, k \neq i, j\}) \\ &= S\Big(T(a,b), T(a,c), T(a,d), \\ &\qquad , T(b,c), T(b,d), T(c,d)\Big) \end{aligned}$$

while the definition of eq. 10 would give:

$$\begin{aligned} T3(a,b,c,d) &= S(T2(b,c,d), T2(a,c,d), \\ &\qquad , T2(a,b,d), T2(a,b,c)) \end{aligned}$$

Expanding this in all terms yields

$$T3 = S(S(T(b,c), T(c,d), T(b,d)), \ldots)$$

and using associativity of the S-norm gives that the expansion contains terms like $S(T(b,c), T(b,c))$. However, only for the standard S-norm maximum we have $S(a, a) = a$ for any $a \in [0, 1]$, so there are values $a, b, c$ and $d$ for which both definitions will not match.

As a result from this discussion, we propose to consider the following definition for a Tk-norm, by not replacing the maximum operator with the S-norm:

**Definition 1** *Given is a set $A$ of $n$ values $a_i$ $(i = 1, \ldots, n)$ from $[0, 1]$, and a T-norm $T$. For $k = 1$ the T1-norm is given by*

$$T1(a_{1,}, \ldots, a_n) = \min(a_1, \ldots, a_n),$$

*for $1 < k \leq n$ the Tk-norm is given by*

$$Tk(A) = \max_i(T(k-1)\{a_j \,|\, j \neq i\})$$

*with $1 \leq i, j \leq n$, and $Tk(A) = 1$ for $k > n$.*

There is a corresponding definition for the Sk-norm:

**Definition 2** *Given is a set $A$ of $n$ values $a_i$ $(i = 1, \ldots, n)$ from $[0, 1]$, and a S-norm $S$. For $k = 1$ the S1-norm is given by*

$$S1(a_{1,}, \ldots, a_n) = \max(a_1, \ldots, a_n),$$

*for $1 < k \leq n$ the Sk-norm is given by*

$$Sk(A) = \min_i(S(k-1)\{a_j \,|\, j \neq i\})$$

*with $1 \leq i, j \leq n$, and $Sk(A) = 0$ for $k > n$.*

Replacing the Tk- or Sk-norms in these definitions by T(k-1)- and S(k-1)-norms, and by using the associativity of the maximum and minimum operator, the recursive definitions transform into the corresponding non-recursive definitions.

## 3 Properties of the Tk-Norm

In this section, we will consider some properties of the Tk-norm of def. 1. The corresponding properties of the Sk-norm can be shown as well.

At first, we start to simplify the definition 1 by using the following theorem:

**Theorem 1** *Given is a set $A = (a_1, \ldots, a_n)$ of real numbers from $[0, 1]$ with $a_1 \leq a_2 \leq \ldots \leq a_n$ and $1 \leq k \leq n$. Then for any choice $B$ of $n - k + 1$ values from $A$ it holds*

$$T(B) \leq T(a_k, a_{k+1}, \ldots, a_n).$$

This can be easily seen by using monotonicity of the T-norm: any other subset $B$ of $(n-k+1)$ elements from $A$ than the one excluding the first $(k-1)$ smallest elements of $A$ will contain only smaller (or equal) elements, thus having a smaller (or equal) T-norm as well. From this, the evaluation of the maximum of all such subsets, as given by def. 1, can be reduced to the evaluation of the T-norm of the subset excluding the first $(k-1)$ smallest elements:

**Theorem 2** *For any set $A = (a_1, a_2, \ldots, a_n)$ with $0 \leq a_1 \leq \ldots \leq a_n \leq 1$, and any T-norm $T$ it holds:*

$$Tk(A) = T(a_k, a_{k+1}, \ldots, a_n).$$

As remarked above, this also gives the corresponding extension of eq. 1 to the fuzzy case.

Using theorem 2, we can show further properties of the Tk-norm easily. Without loss of generality, here we assume the values of $A$ already sorted in increasing order.

**Property 1** *The Tk-norm is bounded by 1:*

$$Tk(a_1, \ldots, a_n, 1) = Tk(a_1, \ldots, a_n)$$

This follows from

$$\begin{aligned} Tk(a_1, \ldots, a_n, 1) &= T(a_k, \ldots, a_n, 1) \\ &= T(a_k, \ldots, a_n) \\ &= Tk(a_1, \ldots, a_n) \end{aligned}$$

**Property 2** *The Tk-norm is monotone in each argument: For any $a_l \leq a_l^*$ it holds*

$$Tk(a_1, \ldots, a_l, \ldots, a_n) \leq Tk(a_1, \ldots, a_l^*, \ldots, a_n).$$

To see this, three cases have to be considered, depending on the relations of $a_l$ and $a_l^*$ to the $k$-th smallest element. The easy proof will be omitted here.

A remark has to be done on associativity. Usually, set ranking operations are not associative. As an example, consider the sets

$$A = (1, 2, 3, 4, 5, 6, 7, 8, 9)$$

and

$$B = ((1, 2, 3), (4, 5, 6), (7, 8, 9))$$

We can see that $min2(A) = 2$ while

$$min2(min2(1, 2, 3), min2(4, 5, 6), min2(7, 8, 9))$$

computes to $min2(2, 5, 8) = 5$, thus brackets can not be generally removed for the $min2$-operator. So, it is no surprise that the Tk-norm is not associative as well.

It should be remarked that the Tk-norm is obviously commutative.

Another question of interest is how the Tk-norms for different values of $k$ are elated to each other. This will be established by the following theorem.

**Theorem 3** *For any Tk-norm and T-norm it holds*

$$T = T1 \leq T2 \leq \ldots \leq Tn = \max$$

*as well as*

$$Tk \leq mink$$

*for any* $1 \leq k \leq n$.

The proof of the first part of this theorem is as follows:

$$\begin{aligned} Tk(A) &= T(a_k, a_{k+1}, \ldots, a_n) \\ &= T(a_k, T(k+1)(A)) \\ &\leq T(k+1)(A) \end{aligned}$$

since in general $T(a, b) \leq b$, and again assuming the values of $A$ to be sorted. To see that the Tn-norm equals the maximum, theorem 2 gives

$$Tn(A) = T(a_n) = a_n = \max(A)$$

and to see the second part of the theorem, it suffices to consider

$$\begin{aligned} &Tk(A) = T(mink(A), a_{k+1}, \ldots, a_n) \\ &\quad \leq mink(A) \end{aligned}$$

This means that Tk-norm values are numerically distributed in the range of the data itself, no matter how many data there are. Therefore, instead of using a T-norm e.g. to compute the intersection of a larger number of fuzzy sets, an appropriate Tk-norm can be selected and used as well.

# 4 Application of Tk-Norm

As an example, we will consider the application of Tk-norms to define fuzzy morphology operators in image processing, where the direct use of T-norms is restricted to small sizes of the structuring element and can now be increased to any size.

More formally, be an image given by its *grayscale image function*:

$$I : [0, \ldots, w-1] \times [0, \ldots, h-1] \rightarrow [0, \ldots, g_{max}]$$

with $w$ the width of the image, $h$ the height and $g_{max}$ the maximum intensity value (255 is a common value). A tupel $(x, y, I(x, y))$ is called a pixel. A set of offsets $U = \{(i, j)\}$ assigns a local neighborhood to each pixel $(x, y)$ by $I_U = \{(I(x+i, y+j))\}$. If for all $i$ and $j$ holds $i, j \in \{-1, 0, 1\}$ this gives the so-called 8-neighborhood. Then, we define the image operator Tk-norm by the new image function:

$$Tk \circ I(x, y) = Tk\left(\frac{I_U(x, y)}{g_{max}}\right) \cdot g_{max}. \quad (13)$$

Figs. 1 and 2 give some results of such an operation. In fig. 1, the brightness increment of

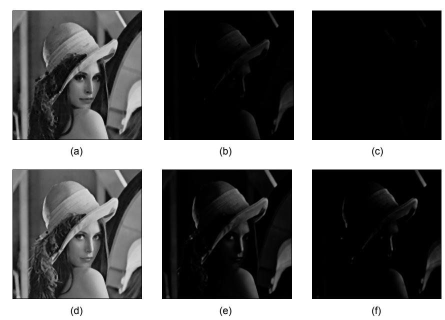


Figure 1: T-norms (upper row) compared to T5-norms (lower row): (a)+(d) Minimum; (b)+(e) Product norm; (c)+(f) Frank norm ($\lambda = 100$). The norms were computed in a local $3 \times 3$ neighborhood.

using Tk-norms instead of T-norms is demonstrated. Using T-norms like the FRANK-norm in such a context (subfigure 1(c)) seems useless, since the result image is nearly black. Once using Tk-norms with higher values, the operator starts to provide brighter values. Figure 2 shows the sequence of all Tk-norms for $k = 1, 2, \ldots, 9$.

# 5 Conclusion

The advantage of using Tk-norms instead of T-norms has been demonstrated by the provision of a simple image processing operator. However, other tasks like providing training rules for neuro-fuzzy networks, specification of fuzzy


Figure 2: Tk-norms computed for Lena image, using Dubois-Prade T-norm with $\alpha = 0.9$. The norms were computed in a local $3 \times 3$ neighborhood.

fusion operations, fuzzy selection schemes or defuzzification rules in fuzzy control are among the potential applications of the proposed Tk- and Sk-norm approach as well.

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