Relational Approaches to Resource-Aware Multi-Maxmin Fairness in Multi-Valued Resource Sharing Tasks

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Abstract: Here we study formal approaches to maxmin fairness in multi-valued evaluations. In such resource sharing or distribution tasks among agents, each share has multiple evaluations according to different observables. Maxmin fairness cannot be directly applied in this situation. To allow handling of fairness aspects in such circumstances as well, we promote a strict relational framework. Maxmin fairness will be specified as a set-theoretic relation in two alternative ways, thus generalizing maxmin fairness for multi-vectors. In addition, the relation will be restricted by an additional resource fairness criterion. For an example case of multi-resource allocations, the higher utilization of resources will be demonstrated.

Keywords: fairness; maxmin fairness; preference modeling; multi-fairness; fair multi-resource allocation

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1 Introduction

In multi-valued evaluations we assume that a number of agents all evaluate a state by multiple means. This can appear in a resource sharing task, where each agent valuates his or her share under various aspects, or in a multi-resource sharing job scheduling, where each job consumes a number of resources varying for each particular job-issuing user. The basic representation here is a multi-valued vector (or multi-vector) where each component is a vector itself, expressing the various evaluation values of a resource allocation in an ordered manner. We can give two examples for such a representation. At first, assume a multi-resource sharing in a job scheduling. For example, a datacenter admits customer jobs that consume CPU and memory. For each customer, the number of CPUs and amount of memory consumed by each job might differ. Within a time frame, a total number of CPUs and a total amount of memory (the resources) are available to a number of customers. An allocation is the assignment of the number of jobs (not necessarily an integer number) to each customer that they can perform. These numbers of jobs determine the CPU and memory needs of each customer. So a component of the multi-vector is a vector of two components: CPU usage and memory usage. At second, we might consider a single resource sharing, but under different value aspects. For example, in the specification of a routing in end-to-end user traffic in a data network, the same routing gives raise to different evaluations like delay time, number of hops, and traffic rates and thus, each component of a multi-vector would represent the agents' particular evaluations and be a vector of three components. Note that the first and second aspect cannot be combined. For example, delay time cannot be modeled as a sharable resource. In general, we can also consider a mixed representation, where each single resource is evaluated in a resource-dependent number of ways.

The decision problem here is to find a suitable allocation, with regard to some criterions of optimality, efficiency, or fairness. Obviously this will much depend on the manner in which these criterions are specified. Various notions of fairness have been provided, starting with fairness measures like Jain's fairness index proposed in Jain et al. (1984), maxmin fairness, and following Kelly et al. (1998) proportional fairness, α-fairness etc. Fairness is often studied within a close application context, for example in Dianati et al. (2005) an utility-based framework to evaluate the degree of fairness of resource allocation schemes in wireless access networks is proposed, Bredel & Fidler (2009) studies a fairness model of an ideal fair queuing system and its realization through the Distributed Coordination Function in IEEE 802.11, Srinivasan & Somani (2003) presents a protocol for near-optimal channel utilization, circumventing unfair delay characteristics, and Uchida & Kurose (2009) discusses information theoretic aspects of fairness. More on the tradeoff between fairness and efficiency can be found in Tang et al. (2006) and Zukerman et al. (2005), while further sharing objectives like potential delay minimization were introduced in Massoulié & Roberts (2002). Guarantees of fair bandwidth allocations were the topic of Bonald & Massoulié (2001). These studies on fairness were focusing on single resource usage only, and extensions to multi-valued or multi-resource problems are not straightforward.

Recently, Dominant Resource Fairness (DRF) has been proposed in Ghodsi et al. (2011a) and Ghodsi et al. (2011b). There, *maxmin fairness* is extended to the case of distribution of several resources like CPU time and memory on a per-job basis at the same time. For each agent (task), the maximal share of a resource is selected, and *maxmin fairness* is sought for these so-called dominant shares. It corresponds to a mapping of a multi-valued vector to a standard vector, where the maximum shares replace the corresponding evaluation vectors. The method is compared to the popular asset fairness and the Competitive Equilibrium from Equal Incomes (CEEI) approach of microeconomics theory (see Varian (1974), Young (1994), Moulin (2004)) by a catalogue of criteria for such resource sharing regimes. In asset fairness, an equal valued share for each resource has to be identified. Then, assigning equal shares to each agent is formulated as a combinatorial optimization problem. In CEEI, all n agents start from an equal share of $1/n$ and then start to perform mutual transactions in order to achieve a Nash equilibrium.

Dominant Resource Fairness appears to be an attractive solution for multi-resource allocation, esp. with regard to its simple linear-complexity allocation algorithm (following the pattern of the bottleneck flow control algorithm) and fulfilling several fairness-relevant properties like sharing incentive, strategy-proofness, envy-freeness, Pareto efficiency, and more. The approach found a continuation in Joe-Wong et al. (2012), where the whole multi-valued vector that represents an allocation is fused into a single *fairness measure* value. The goal is to find an allocation that maximizes such a fairness measure. The composition of such fairness measures follows an axiomatic theory of fairness as it was considered in Lan et al. (2010) and provides a number of parameters of a general expression. The paper proposes the use of the same function expression for dominant resources (so-called Fairness on Dominant Shares, FDS) or jobs (so-called Generalized Fairness on Jobs, GFJ).

This function expression has the general shape

$$F_{\beta,\lambda}(x)= \operatorname{sgn}(1-\beta) \times \\ \times \left(\sum_{i=1}^{n} \left(\frac{\mu_i x_i}{\sum_{k=1}^{n} \mu_k x_k} \right)^{1-\beta} \right)^{\frac{1}{\beta}} \left(\sum_{i=1}^{n} \mu_i x_i \right)^{\lambda}$$

which depends on two parameters β and λ as well as n weights μ_i to compute the "amount of fairness" of an allocation expressed by vector x. The influence of parameter choices on the fairness-efficiency tradeoff is intensively investigated in Joe-Wong et al. (2012).

We give a critical discussion of three special aspects, where these approaches show weaknesses:

1. All approaches focus on the fusion (or aggregation) of multiple informations about the agents evaluations into single numeric values. These values then represent "states of wealth" that can be used for direct comparison between the outcomes of different allocations. This is a very convenient way, comparable to the common method of averaging to compare two series of data. However, single values cannot represent mutual relations between the single components of multi-vectors. Moreover, we miss an important aspect of subjective decision making, for example expressed in the seminal paper of Kahneman and Tversky on prospect theory (Kahneman & Tversky (1984)) "People

do not normally think of relatively small outcomes in terms of states of wealth but rather in terms of gains, losses, and neutral outcomes (such as the maintenance of the status quo).” Comparison of gains and losses followed by a preference decision is a pair-wise relation, and it might not always be possible to reduce this to a numerical comparison. Achieving neutrality in comparison is even impossible.

2. There is no explicit reference to the total resource usage. The customer is represented by the maximum demand of a resource, while no information about the other resources is involved. Thus, it also cannot be judged on the total resource utilization of an allocation, taking only maximal demands into account. Therefore, the additional investigations into the fairness-efficiency tradeoff were necessary.

3. These approaches can only be applied to exhaustible resources. Where there are just different aspects of a resource sharing, as for example for delay time, “dominant resource” does not make sense.

In this paper, we will provide a framework to handle all these aspects in a natural and flexible manner. In particular, we promote the use of a strict set-theoretic relation based framework to represent maxmin fairness, and from this gain access to a formal treatment of fairness within multi-valued evaluations. By adding sufficient constraints to the relations, it becomes possible to incorporate the aspect of resource utilization, as well as non-sharable allocation aspects (like delay). Moreover, the proposed framework will include DRF as well as FDS and GFJ as special cases, for the latter two at least in theory.

The use of relational mathematics in fair division problems is more common. In Bouveret & Lang (2005), authors study a *fair division problem of indivisible goods without money transfers*. Here, in addition to the agents and goods, to each agent a preference relation is assigned. The fair division problem then is formulated as the task to achieve Pareto-efficiency and envy-freeness. The innovation in this work is the transition to propositional logic and formal language theory, and also allows for example for analysis of complexity of the fair division problem. Here, we will follow the concept that each agent valuates by his or her own preference relation and then extend Pareto efficiency to *multi-maxmin fairness* in the same way as Pareto-equilibrum is extended to the *maxmin fair state* by giving an extra focus on agents that already receive less than other agents. Thus we assume that it is possible to handle the accompanying problems within the framework of relational mathematics alone.

In Section 2 we summarize relational frameworks that will be used to reconsider maxmin fairness in Section 3. Section 4 applies the relational framework to maxmin fairness and provides corresponding definitions. Then, section 5 will provide a study on the application of the concept to a multi-resource sharing study problem.

The main contributions of present work can be seen in the formal specification of a multi-fairness, directly following *maxmin fairness* (and including *maxmin fairness* as a special case) with design flexibility according to specification of agent relations, and the demonstration of this design flexibility and its implications.

2 Relational Framework

In this section, we will provide the abstract relational framework, without considering a specific relation.

Given a set A, the set-theoretic representation of a *relation* R between elements of A (sometimes called the *domain* of that relation) is given as a subset of $A \times A$. Thus, if an ordered pair (x, y) of elements $x, y \in A$ belongs to R it is understood that “x is in relation R to y.” We will use infix notation for such a relation and write $x \geq_R y$ alternatively to $(x, y) \in R \subseteq A \times A$.

The reason for using this symbol, with reference to the common larger-or-equal relation among real numbers, will become clear in a moment. Just note that elements of relations are *ordered* pairs. It means there is a difference between an element in first position and in second position. If the pair obtained from swapping first and second position does not belong to the same relation, the position is meaningful, and indicates a kind of prevalence of the first element against the second element. This aspect is represented by two well-known properties that relations can take (x and y are general elements of A):

- **Symmetry**: From $(x, y) \in R$ follows $(y, x) \in R$.
- **Asymmetry**: From $(x, y) \in R$ follows $(y, x) \notin R$.

Then, we can decompose any relation into a symmetric and asymmetric part.

Definition 1 *Given a relation R over domain A. The asymmetric part $P(R)$ of R is a relation over the same domain A which is given point-wise as*

$$(x, y) \in P(R) \longleftrightarrow (x, y) \in R \wedge (y, x) \notin R \qquad (1)$$

It appears that $R - P(R)$ is a symmetric relation, as it contains all pairs (x, y) of R where (y, x) also belongs to R (this might include reflexive cases $(x, x) \in R$).

Definition 2 *Given a relation R over domain A. The symmetric part $I(R)$ of R is a relation over same domain A which is given point-wise as*

$$(x, y) \in I(R) \longleftrightarrow (x, y) \in R \wedge (y, x) \in R \qquad (2)$$

Then, generally $P(R) \cap I(R) = \emptyset$ and $P(R) \cup I(R) = R$.

Now, among all possible relations over a given domain A, we can define an equivalence between relations R_1 and R_2 in case $P(R_1) = P(R_2)$. It can be easily seen that this is an equivalence relation (with the domain being the set of all relations with same domain A), and thus, the set of relations can be decomposed into equivalence classes that are disjoint and whose union is the set of all these relations. Each such class contains a single asymmetric relation, the shared asymmetric part of each relation within a class. For this relation, we will use the notation $x >_R y$. The only exception is the class where the only asymmetric relation is the empty set (this class contains all equivalence relations, but also other symmetric relations), here we do not have any comparison aspect.

Example: Given set $A = \{a, b, c\}$ and a relation $R = \{(a, a), (a, b), (b, a), (c, b), (c, c)\}$. We see that $P(R) = \{(c, b)\}$ and $I(R) = \{(a, a), (a, b), (b, a), (c, c)\}$, $P(R)$ and $I(R)$ are disjoint and especially $R = P(R) \cup I(R)$. The relation $R' = \{(b, b), (c, b)\}$ belongs to the same class as R since it has the same asymmetric part $P(R') = \{(c, b)\}$.

Now, for any relation R we can consider extreme elements. At first, we have to restrict R to its asymmetric part, and then define (using infix notation):

Definition 3 *Given a relation R between elements of a domain A. Then, the maximum set $M(R)$ of R is the set of all $x' \in A$ such that there is no $y \in A$ with $y >_R x'$ (or: no $y \in A$ with $(y, x) \in P(R)$).*

It appears that maximality is solely defined on a set-theoretic base, and no reference to other mathematical structures is needed. As soon as we have a relation and a domain, there are also maximal elements.

However, such maximum sets can be empty (for example, for symmetric relations). The question if there is a sufficient condition ensuring non-empty maximum sets is answered by the following theorem. We define a relation as *cycle-free* (or *outbound*) if for any subset $\{x_1, x_2, \ldots, x_k\}$ of elements from A from $x_1 >_R x_2 \wedge x_2 >_R x_3 \wedge \cdots \wedge x_{k-1} >_R x_k$ follows $x_k \not> x_1$ (note that we refer to the asymmetric relation here, using the $>_R$-symbol). Then:

Theorem 1 *From R with finite domain being cycle-free (and $R \neq \emptyset$) follows $M(R) \neq \emptyset$.*

For a proof, see e.g. Suzumura (2009), Theorem $A(3)$. This property of cycle-free relations is also base for a general *ranking* of elements of A by R. Intuitively, we take the maximum set of A as elements of rank 1. Then, we consider $A_1 = A - M(R)$. The relation R restricted to this subset of A must be cycle-free as well, and we can find its non-empty maximum set, giving elements of rank 2. This can be continued until all elements are ranked.

Example: Take same set $A = \{a, b, c\}$ and relation $R = \{(a, a), (a, b), (b, a), (c, b), (c, c)\}$ as in the example before. From $P(R) = \{(c, b)\}$ follows that a and c both are maximal elements (no element of A is in relation R to them). Removing a and c from A gives the set $A' = \{b\}$ and R restricted to this set (i.e. not considering any pair containing a or c in either position) is $R_{|A'} = \emptyset$. Then, its asymmetric part is the empty relation as well, and no element is in relation to b. Thus, b is maximal in A' and of rank 2.

We conclude this section by taking note of the fact that all definitions so far are based on elementary set theoretic concepts. Thus, having an arbitrary relation, no matter over which domain, we also have a concept of *asymmetric part*, *maximality*, the potential property of *cycle-freeness* and *ranking*. Thus, the relational framework, given any (symbolic or numeric) domain A and corresponding relations, allows to assess a meaning of “optimality” by considering maximal elements of (cycle-free) relations.

3 Maxmin Fairness

We may now consider more relevant examples of relations, especially relations with domain R^n (sometimes called vector-relations, or vector inequalities). The simplest way to expand the concept of “$\geq$” from R^1 to R^n is the Pareto-dominance relation. We can understand it as n-fold application of element-wise comparison by the real-valued $\geq$-relation.

Definition 4 *Given two vectors x and y from R^n with components x_i and y_i respectively. Then $x \geq_P y$ iff for all $i = 1, \ldots, n$ it holds that $x_i \geq y_i$.*

Remark: Note that the common definition of Pareto-dominance requires also at least one component of x to be truly larger than the corresponding component of y. However, by using the framework presented in the foregoing section, we do not need to make this comparison. This relation is simply the asymmetric part of the Pareto dominance relation. From transitivity of the real-valued size relation, cycle-freeness of Pareto dominance follows easily.

For fair division of sharable goods, *maxmin fairness* is defined as a state where no agent can be made better off without making another already equally or worse off agent even more worse off. *Maxmin fairness* has been shown to maximally utilize resource sharing, for example for end-to-end user traffic rates in a communication network, where the *maxmin fair state* can be achieved by the Bottleneck Flow Control algorithm, originally proposed in Bertsekas & Gallager (1992).

Definition 5 *Given a feasible space $A \subseteq R_n$. For two elements (vectors) x and y from A it is said that x **maxmin fair dominates** y ($x >_{mmf} y$) iff for each i with $y_i > x_i$ there is at least one $j \neq i$ such that (1) $x_i \geq x_j$ and (2) $x_j > y_j$.*

Technically, this definition appears rather inconvenient. We can specify a test of maxmin fairness in a more simple form, following these steps (we will call it *Algorithm-I*):

1. For a pair (x, y) derive the set S of indices from $\{1, \ldots, n\}$ where $x_i \neq y_i$.
2. Find the minimum value $m = \min_S x_i$ among all $i \in S$.
3. Select the index set $I = \{i \,|\, i \in S \wedge x_i = m\}$.
4. $x \geq_{mmf} y$ iff $\min_I y_i < \min_I x_i$.

Example: Figure 1 gives an example for the evaluation of the case $x = \{3, 4, 3, 3, 6\}$ and $y = \{3, 6, 2, 4, 5\}$. Both vectors differ in the components $S = \{2, 3, 4, 5\}$ and the smallest x-component among these is 3. Thus, $I = \{3, 4\}$

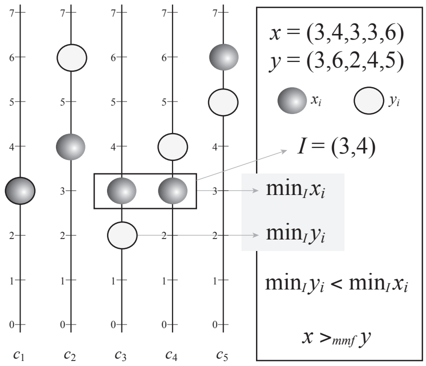

Figure 1 Alternative test for maxmin fairness relation.

and we see that the minimum value of y-components with index from I is $2 < 3$. Thus, $x \geq_{mmf} y$.

Remark: Note that this way, the naming of maxmin can become more clear. But it has to be noted that this is not the same as maximizing the minimum (as e.g. compared to the minmax strategy in game theory). Reason is that the comparison is referring to differing components only.

We can convince ourself that both, Def. 5 and *Algorithm-I* represent the same relation. (1) Def. 5 $\rightarrow$ *Algorithm-I*: Assume the opposite, $x \geq_{mmf} y$ but for all $i \in I$ $x_i \leq y_i$. However, $x_i = y_i$ is not possible since $I \subseteq S$ (elements of I were selected among indizes, where corresponding components of x and y differ), thus $y_i > x_i$. According to the definition of maxmin fairness, then in each case $i \in I$ there must be a j such that $x_j \leq x_i$ and $x_j > y_j$. But x_i is already the minimum value among all components with indices from S and therefore $x_i = x_j$ or $j \in I$. Then $x_j > y_j$ is in conflict to the assumption $x_i < y_i$. (2) *Algorithm-I* $\rightarrow$ Def. 5: We have at least one index j where $y_j < m$. Now for any i with $y_i > x_i$ (and thus $i \in S$) this j has both properties: $x_j \leq x_i$ since $x_j = m$ and $x_j = m > y_j$. Thus, both definition and algorithm are equivalent.

Going a little bit further, we note that permuting the components of both vectors x and y in the same way does not affect the definition of maxmin fairness (neither does it *Algorithm-I*). Then we can also rewrite $Algorithm - I$ as:

1. Sort the components of x in non-decreasing order.
2. Sort the components of y in the same order as x in the foregoing step.
3. If and only if the sorted vector x lexicographically comes after the sorted y then $x \geq_{mmf} y$.

Example: Take again $x = \{3, 4, 3, 3, 6\}$ and $y = \{3, 6, 2, 4, 5\}$. The components of x in sorted order are $\{3, 3, 3, 4, 6\}$. A permutation giving this order would be (13425) (due to multiple occurrence of 3, there are other ways as well but this will not affect the procedure). Applying this permutation to y gives $\{3, 2, 4, 6, 5\}$ which comes lexicographically after sorted x (first component is the same, but the second is smaller).

We note that maxmin fairness is an antisymmetric relation, i.e. from $x \geq_{mmf} y$ and $y \geq_{mmf} x$ follows $x = y$. Thus, the asymmetric part of R_{mmf} is obtained by removing all pairs (x, x). Also foregoing discussion shows that from $x \geq_{mmf} y$ follows that $\min_i x_i \geq \min_i y_i$. However, the case of equality makes the validation of cycle-freeness rather complex. We shifted the proof to the Appendix.

As last point we want to comment on the relation of Dominant Resource Fairness to the maxmin fairness relation. In Ghodsi et al. (2011a) DRF has been provided as an allocation algorithm, thus also here it is not primarily given as maximal element of a relation. However, the relation to maxmin fairness is straightforward. If agent i consumes the share μ_{ij} of the available resource j then we assign an index i' to each agent, where $\mu_{i'j}$ is maximal. This is the dominant share of agent i. Then, an allocation of resources to agents is characterized by the vector of all dominant shares, and DRF corresponds with the maximum set of the maxmin fairness relation for the vector of dominant shares. Without going into detail we note that this is not the only relation serving the DRF allocation, for example, the leximin relation between dominant shares will give the same result.

4 Multi-Maxmin Fairness Relations

We consider a multi-objective distribution problem, where an operator distributes a number of items (their number does not matter here) to a number n of agents (or users). A specific distribution Δ is characterized by m measurements and thus, it is represented by assigning an m-dimensional *objective vector* to each agent. By $(x_i)[\Delta]$ we denote the objective vector assigned to agent i. Its components, i.e. the single measurement values for each agent, are indicated by $(x_i)_j[\Delta]$ (we will not give the argument Δ if it is clear from context). For example

$$X = \left(\begin{pmatrix} 8 \\ 6 \\ 9 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 8 \end{pmatrix} \right)$$

represents an allocation of items to 3 agents, where for each agent the quality of the allocation is represented by three, possibly competing, measured objectives. So, the 3rd measurements qualifies the allocation represented by this multi-vector to agent 1 by the value 9, the highest among the measurements among all agents.

In order to compare two such multi-vectors, and thus specifying a set-theoretic relation between allocations, we propose a two step procedure. In the first step, we compare for resource fairness. It means that for an allocation of resources, we aggregate the single agent allocations into a single value (typically by taking the sum, or the maximum in case the objective refers to a non-sharable aspect like delay time) for each objective (resource). Then we obtain two vectors of resource aggregation values, one for each comparison multi-vector. The first requirement is that these resource aggregation vectors have to be in a maxmin fairness relation. The second requirement refers to a generalization of the maxmin fairness relation for the multi-vectors themselves that will be detailed in the following. We say that the two multi-vectors are in multi-maxmin fairness relation (MMMF) to each other if both of these conditions are fulfilled.

By the first condition, we also ensure that this MMMF relation is cycle-free and thus guarantees a ranking among the elements of each finite set of multi-vectors. Otherwise, a cycle of the MMMF relation would imply a cycle of the maxmin fairness relation that is used to compare the resource aggregation vectors. This is not possible (we again refer to the Appendix for the proof).

Now we have to explain how to generalize maxmin fairness to the case of multi-vectors. We will present a rather natural way to extend mamxmin fairness in a manner that (1) such multiple observables can be taken into account, and (2) each agent can follow his or her own interpretation of "better off" and "worse off." For doing so, we start with an observation on the formal structure of Definition 5.

There, the use of $>, \geq$ is not confined to the use of the same relation. Essentially, three relations are taken into account when justifying *maxmin fair dominance*:

$$\begin{tikzcd}
x_1 & x_2 & \cdots & x_i \arrow[rr, Rightarrow, "\geq"] & \cdots & x_j \arrow[d, ">"] & \cdots & x_n \\
y_1 & y_2 & \cdots & y_i \arrow[u, ">"] & \cdots & y_j & \cdots & y_n
\end{tikzcd}$$

These relations are between y_i and x_i (a possible improvement for agent i), x_j and x_i (agent j, who is already equally or worse off compared to agent i) and x_j and y_j (again, agent j, who is made even more worse off, and thus would becomes an "envy" in state y). So, one relation is according to the model of agent i, and two according to the model of agent j. But there are three more relations, ensured by transitivity of the larger relation for real numbers: $x_i > y_j, y_i > y_j$ and $y_i > x_j$. These three are also seen from the perspective of agent j.

If we now replace the real number relations by "agent-specific" transitive vector relations $>_i$ and $>_j$, we get the following scheme:

$$\begin{tikzcd}
x_1 & x_2 & \cdots & x_i \arrow[rr, Rightarrow, "\geq_j"] & \cdots & x_j \arrow[d, ">_j"] & \cdots & x_n \\
y_1 & y_2 & \cdots & y_i \arrow[u, ">_i"] \arrow[urr, ">_j"] & \cdots & y_j & \cdots & y_n
\end{tikzcd}$$

Here, we have four relations that exactly corresponds to the relations of former scheme, and two others follow by transitivity (of $>_j$-relation): $y_i >_j y_j$ and $x_i >_j y_j$. Now we can define multi-maxmin fairness:

Definition 6 *For two given multi-vectors X and Y of same dimension with corresponding objective vectors (x_i) and (y_i), we say that X is multi-maxmin fair dominating Y ($X \geq_{mf} Y$) iff for each i with $(y_i) >_i (x_i)$ there is at least one $j \neq i$ with the following properties:*

1. $(x_i) \geq_j (x_j)$

2. $(y_i) >_j (x_j)$

3. $(x_j) >_j (y_j)$

Note that in some cases, the test for the second condition can be skipped, for example if all agents use the same transitive preference relation. Then, in case all agents use the maximum component of their evaluation vectors to compare with other evaluations, this definition gives Dominant Resource Fairness as a special case (when skipping above step 1, i.e. the resource aggregation test).

However, in the foregoing section we provided an alternative definition for maxmin fairness by *Algorithm-I*. This can be used as a base to specify another multi-vector version of maxmin fairness. In this case it will lead to a different relation, and also, in addition to the agent relations R_i we will also need a "master relation" R_0 which is at least cycle-free (so it can be Pareto dominance, maxmin fairness, proportional fairness, or leximin). Then we take advantage of the fact that R_0 allows to rank the vectors composing the multi-vector x. We proceed in analogy to *Algorithm-I* and specify an alternative multi-maxmin fairness R_{mmmf_I} as follows:

1. Determine the least ranked components of multi-vector X according to relation R_0. Their indices comprise the index set S.

2. Determine the subset of indices I of S where vector-components of multi-vector X and Y differ.

3. If and only if for at least one $i \in I$ $(x_i) >_i (y_i)$ then $X \geq_{mmmf_I} y$.

Note that in case of a single objective, i.e. $m = 1$ and if we use the real-valued "$\geq$" relation for all agents as well as master relation, this algorithm is equivalent to *Algorithm-I* and thus R_{mmmf_I} is equivalent to maxmin fairness.

In the following, we will only consider maxmin fairness as master relation R_0.

Now we have two versions of the MMMF relation, further qualified by various choices for the agent relations. In the following, we will restrict this to some relevant cases and introduce a suitable nomenclature. A MMMF relation will be specified by the pattern "Ab" where A is either "I" or "P": "I" means that we use the MMMF relation based on maxmin fairness tested by Algorithm I, and "P" for maxmin fairness according to Def. 5 (i.e. in predicate form, therefore "P"). Then, "b" refers to the preference relation used by all agents. In particular we consider "p" for Pareto-dominance, "t" where all agents compare (evaluation) vectors by the total sum of all components, "i" if agent i compares by the i-th objective, and "m" where all agents compare by the maximum component. Then, "Pm" for example expresses DRF with the additional test of resource aggregation fairness: MMMF is tested by (1) maxmin fairness for the resource aggregation vectors, and (2) generalized maxmin fairness according to Def. 6, using the comparison of maximum components as relations $>_i$.

Example: Consider two multi-vectors X and Y, where agent i is focussing on the i-th objective alone (i.e. relation MMMF-Pi):

$$X = \left(\begin{pmatrix} 3^* \\ 7 \\ 10 \end{pmatrix} \begin{pmatrix} 10 \\ 9^* \\ 10 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \\ 3^* \end{pmatrix} \right)$$

$$Y = \left(\begin{pmatrix} 2^* \\ 5 \\ 6 \end{pmatrix} \begin{pmatrix} 8 \\ 2^* \\ 8 \end{pmatrix} \begin{pmatrix} 5 \\ 8 \\ 5^* \end{pmatrix} \right)$$

At first we test maxmin fairness for the resource aggregation vectors. For the first multi-vector, this gives the vector $(18, 19, 23)$, $(15, 15, 19)$ for the second. The first even Pareto-dominates the second, and this implies maxmin fairness among them. Thus, the first condition is fulfilled. Now we test for the generalized maxmin fairness. By $>_{m(i)}$ we denote a comparison by the i-th component (these components are labeled by a star * above). We see that for agents 1 and 2, there is no improvement in Y: neither $(y_1) >_{m(1)} (x_1)$ (i.e. $(y_1)_1 > (x_1)_1$ or $2 > 3$) nor $(y_2) >_{m(2)} (x_2)$ (i.e. $(y_2)_2 > (x_2)_2$ or $2 > 9$) holds. But for $i = 3$, there is an improvement: $(y_3) >_{m(3)} (x_3)$ (i.e. $(y_3)_3 > (x_3)_3$ or $5 > 3$). So, we have to look if there is a $j \neq i$ fulfilling properties (1) to (3) of Definition 6. The choices for j are 1 or 2. In fact, $j = 1$ has these properties: (1) $(x_3) \geq_{m(1)} (x_1)$ (i.e. $(x_3)_1 \geq (x_1)_1$ or $5 \geq 3$); (2) $(y_3) >_{m(1)} (x_1)$ (i.e. $(y_3)_1 > (x_1)_1$ or $5 > 3$); and (3) $(x_1) >_{m(1)} (y_1)$ (i.e. $(x_1)_1 > (y_1)_1$ or $3 > 2$). So, for this example, agent 1 is "envy" about the potential improvement for agent 3, or $X >_{mf} Y$.

Finally we want to remark that between all possible instantiations of MMMF there can be various dependencies. For example, MMMF-Pp will imply MMMF-Pm as well as MMMF-Pt (if all components of a vector are larger then the corresponding components of another vector, then also sum and maximum of all components will be larger). Then, the maximum set of MMMF-Pp will be a superset of the MMMF-Pt and MMMF-Pm relations.

5 Application to Multi-Resource Sharing

In this section, we discuss the concept of multi-maxmin fairness for a specific model multi-resource sharing task. At first, we consider the same example that was used in Ghodsi et al. (2011a) and Joe-Wong et al. (2012). A datacenter has to process jobs for two customers. For one job, customer 1 needs 1 CPU and 4 GB RAM and customer 2 needs 3 CPUs and 1 GB RAM. In total, 9 CPUs and 18 GB RAM are available. Jobs have to be allocated to each customer in a fair but also efficient manner.

The starting point, like it is taken in Ghodsi et al. (2011a) is to see after the relative resource consumptions. For one job, customer 1 consumes $1/9$ of the available CPUs and $4/18 = 2/9$ of the available RAM. Thus, RAM is the *dominant resource* for customer 1. For customer 2, it is CPU: one job consumes $3/9 = 1/3$ of the available CPUs and $1/18$ of the available RAM.

The idea behind Dominant Resource Fairness (DRF) is to allocate jobs such that the increase of a dominant resource for one customer would result into a necessary decrease of the dominant resource allocated to another customer who has already same or less. It means to seek maxmin fairness in dominant resource consumption, while ignoring all other resource consumptions. For this example, the solution is to allocate 3 jobs to customer 1 and 2 jobs to customer 2. In this case, customer 1 consumes $2/3$ of the available RAM (and $1/3$ of the available CPUs) and customer 2 $2/3$ of available CPUs (and $1/9$ of the available RAM). Thus, each customer would receive $2/3$ of her or his dominant resource share, while in that moment the resource CPU is fully consumed: $1/3$ is allocated to customer 1 and $2/3$ to customer 2. Any attempt to increase a dominant resource share would cause a decrease of the dominant resource share of the other.

Note that this solution can be easily found in a systematic way, following a procedure similar to Bottleneck Flow Control. In Ghodsi et al. (2011a), the approach is extended to the case where some actual job demands are possibly lower and remaining resources are allocated among other customers, using the same algorithm iteratively. Also, in general, the number of jobs is not necessarily an integer, i.e. fractional job allocations are assumed to be possible.

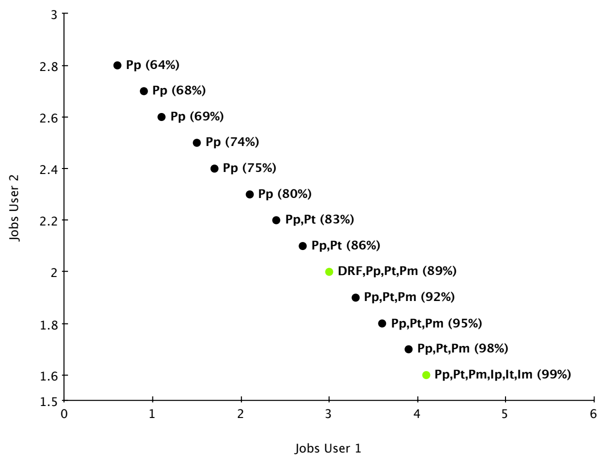

Figure 2 Maximum sets of example problem for various multi-maxmin fairness relations. The values in brackets indicate the total resource allocations. Green dots indicate where the concepts of optimality in Dominant Resource Fairness and Multi-Maxmin Fairness differ.

If we want to solve the same example problem in the presented relational framework, we have to express pairwise comparison between allocations by a multi-maxmin fairness relation, including the specification of the domain of the relation. For example, we can sample all possible values of the number of jobs allocated to customer 1, indicated by α_1 between 0 and the maximum possible value 4.5 (where the resource RAM is fully given to customer 1) in steps of 0.1, and the number of jobs α_2 given to customer 2 between 0 and 3 (where all CPUs are given to customer 2) in steps of 0.1 as well. All feasible allocations by pairs (α_1, α_2) constitute the domain of the relation.

Figure 2 shows maximum sets for various multi-maxmin fairness relations. Here, we used multi-maxmin fairness in predicate and in algorithm I form, where each agent has the preference relation Pareto dominance, total sum or maximum component. In all cases, the aggregation was done by summing up the relative resources and comparison was done by using maxmin fairness. The maximum sets were found by exhaustive search, i.e. comparing each allocation with each other and selecting the allocations to which no other is in relation (belonging to the asymmetric part of the relation, to be precise). We can see that multi-maxmin fairness given in predicate form and where all agents use Pareto dominance as their preference relation (Pp) provides the largest maximum set, containing the maximum sets of all other relations, including the DRF point $(3, 2)$.

On the other hand, multi-maxmin fairness in algorithm I notation, no matter which preference relation is used, all have maximum sets with one element, which is the allocation of 4.1 jobs to customer 1 and 1.6 jobs to customer 2. This point is included in the maximum sets for all multi-maxmin fairness relations, so it seems to have some relevance. Here we show the related allocations (rounded values). For the DRF point we have

$$A(3,2) = \left(\begin{pmatrix} 0.33 \\ 0.66 \end{pmatrix} \begin{pmatrix} 0.66 \\ 0.11 \end{pmatrix} \right)$$

and for the MMMF point (MMMF should mean selected by any multi-maxmin fairness):

$$A(4.1,1.6) = \left(\begin{pmatrix} 0.455 \\ 0.911 \end{pmatrix} \begin{pmatrix} 0.533 \\ 0.088 \end{pmatrix} \right).$$

If we compare the job allocations $A_{DRF} = (3, 2)$ and $A_{MMMF} = (4.1, 1.6)$ we can see a number of advantages of the allocation $(4.1, 1.6)$.

At first, in the total resource allocation for A_{MMMF} the available resources CPU AND RAM are nearly completely allocated to customers (99%) where the DRF allocation leaves 11% of the resources unallocated. Also the total number of jobs is larger for the MMMF allocation: 5.7 versus 5 for DRF.

Second, this is achieved by abandoning the principle to allocate the dominant resource of each customer in a maxmin fair manner. However, it happens to a degree where the disadvantage for the loosing party is smaller than the gain for the winning party. Customer 2 receives 20% less jobs (2 to 1.6), while customer 1 receives more than 30% more jobs (3 to 4.1).

Third point: since $(3, 2)$ does not belong to maximum sets of multi-maxmin fairness in algorithm I notion, there must be allocations in relation to it. For example, the allocation $(4.0, 1.6)$ is in relation Ip to the allocation $(3, 2)$. The multi-vector of the former allocation is

$$A(4,1.6) = \left(\begin{pmatrix} 0.44 \\ 0.88 \end{pmatrix} \begin{pmatrix} 0.53 \\ 0.09 \end{pmatrix} \right).$$

The total for each resource gives the vector $(0.97, 0.97)$ and this maxmin fair dominates the same aggregation for the DRF point with $(1.0, 0.77)$: the increase in first resource CPU from 0.97 to 1 in the DRF allocation is accompanied by a decrease from same or less 0.97 to 0.77 for the second resource RAM. So it fulfills the first condition of multi-maxmin fairness. For the second condition, we have to rank the component vectors of $A(4, 1.6)$ by maxmin fairness. The vectors are not in relation to each other, so both have the rank 1, which is then also the least rank. One of them, $(0.44, 0.88)$ Pareto-dominates the corresponding vector $(0.33, 0.66)$ of the DRF allocation. Thus, also the second condition is fulfilled and the DRF allocation $(3, 2)$ is not maximal indeed. It might be noted that the MMMF allocation itself is not in relation to the DRF allocation.

These arguments show that there is good reason to give 4.1 jobs to customer 1 and 1.6 (only) to customer 2, especially to increase the total resource utilization while maintaining fairness.

Now we also conduct an experiment to see how much this example refers to the general case. For the experiment, we consider one more customer, and assume the resource consumptions to be random values between 0 and 1 for each resource (with a granulation of 0.1 for numerical convenience). For each of the MMMF relations, 100 random settings of resource demands have been tested. For each setting, the three job allocations where set between 0 and 10 in steps of 0.1, but only feasible allocations were kept (this gives between 200 and 3000 feasible allocations, depending on the resource demands). Among them, the maximum sets were computed as well as the DRF allocation. Among the maximal elements, the largest total resource utilization was computed.

Table 1 Average largest total resource utilization for maximum sets of 100 random multi-resource problems (Range indicates 20% and 80% quantiles, Ratio means ratio to DRF allocation).

Relation	Average	Range	Direct Ratio
DRF	88.26	82 - 96	1.0
MMMF-Pt	96.25	96 - 100	1.13
MMMF-Pm	97.76	99 - 100	1.11
MMMF-Pp	97.94	99 - 100	1.13
MMMF-It	96.8	98 - 100	1.12
MMMF-Pm	96.27	98 - 100	1.13
MMMF-Pp	96.7	99 - 100	1.13

Table 1 shows some average results for total resource utilization. The first row shows the result for DRF, the following rows for the variants of MMMF. It clearly demonstrates the higher resource awareness of the proposed multi-maxmin fairness. In all cases, maximal elements of multi-maxmin fairness relation achieve 11% to 13% larger resource utilization, if compared to the DRF allocation, and the 100% values of the 80%-quantile show that in at least 20% of the cases the fair allocation is such that the resources are fully consumed. This also includes the case MMMF-Pm which corresponds to DRF enhanced by the resource aggregation test. The fact that the 20%-quantile is larger than the average gives some indication that there are sometimes strong outliers, but nevertheless, there is no single case where a higher resource utilization was achieved by DRF. We also want to mention that between the MMMF themselves there are no significant differences.

6 Summary and Future Work

The extension of *maxmin fairness* to the case of multiple objectives (different observations of the allocation of a shared resource) and multiple agent models (agent-specific relations between the objective vectors) has been introduced. It is first based on aggregate resource fairness as a global test, followed by a maxmin fair component-wise comparison. Maximal elements of the resulting relation then give possible allocations. The approach has been demonstrated for a multi-resource sharing task and it could be clearly shown how the fairness-efficiency tradeoff is handled by this approach, giving fair solutions were nearly or even all resources are consumed. The use of a strict relation-theoretic frameworks makes clear that this does not happen per accident but as a consequence of the systematic design of the relation.

There are many open issues that are subject of further research. A primary target is the extension to larger scenarios. The reason to consider only a rather small number of agents in the examples here was the increased effort to find maximal elements. This effort grows with the square of the size of the relation domain. Here, the approach to use meta-heuristic approaches will be subject of further developments. Independently, the application to other multi-resource sharing tasks where each resource has several evaluations — its way has already been paved in the formal framework that was presented here — has to be studied in more detail.

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Appendix: Proof that maxmin fairness is cycle-free

To study properties of maxmin fairness, we can use a "prototype notation" regarding the fact that the check for maxmin fairness between two vectors x and y is solely based on size comparisons and inequality between real numbers, and not affected by permuting the components of x and y together. Thus, instead of $x \geq_{mmf} y$ we can consider a typical vectors for x, y and their element-wise relations without loosing generality of results. Thus, for vector x we can write

x:	1_+	2_+	3_+	3_+	$\geq 4_+$	$\ldots$

assuming that we have resorted the components of x in non-decreasing order. Then, we replace the x_i by "typical" values like integers $1, 2, \ldots$. Each value can appear one or more times, indicated by the following $+$-sign. Such repetitions can be separated, for example, we have separated the sequence of 3-values above in two groups. Then $\geq 4+$ followed by the dots means that all following elements are all larger or equal to 4.

If we write two or more vectors in tabular form, we indicate component-wise correspondence. So a group of 1+ above another group of 1+ means that the number of repetitions is the same. Now we can specify a prototype notation for $x \geq_{mmf} y$ in case that $x \neq y$ as follows.

index:				i			
x:	1_+	2_+	3_+	3	3_+	$> 3_+$	$\ldots$
y:	1_+	2_+	3_+	< 3	$*$	$*$	$\ldots$

Since $x \geq_{mmf} y$ it follows that there is at least one component $x_i = \min_{x \neq y}$ where $y_i < x_i$. Here, it is the component of x with value 3, in other words $\min_{x \neq y} = 3$. Before this component, x and y have equal components. After this component, x-components must be larger or equal 3, and y-components can be freely chosen (indicated by the $*$).

We can see how this notation works by proving the following lemmas. In the following, $\min_{x \neq y}$ indicates the smallest component of vector x among all components that differ from the corr. component of vector y.

Lemma 1 *If* $x \geq_{mmf} y$ *and* $x \neq y$ *then* $\min_{x \neq y} > \min_{y \neq x}$.

P: We see in above prototype notation that for index i $y_i \neq x_i$ and thus $\min_{y \neq x} \leq y_i < 3 = \min_{x \neq y}$. □

Lemma 2 *If* $\min_{x \neq y} > \min_{y \neq x}$ *and* $y \neq z, y \geq_{mmf} z$ *then also* $\min_{x \neq z} > \min_{z \neq x}$.

P: The fact that $\min_{x \neq y} > \min_{y \neq x}$ can be represented in prototype notation as

index:				i			
x:	1_+	2_+	3_+	$> 3_+$	$\ldots$		
y:	1_+	2_+	3_+	3_+	4_+	5_+	$\ldots$

Here we have sorted the components of y in non-decreasing order. Then, for index i the first time a component of x is different and so $min_{y \neq x} = 3$. We have also sorted such that all components where both, x and y have the value 3 appear before i. Then, all components of x with an index larger or equal i must be larger than 3. Assume a component has a value a smaller or equal 3, then for sure it will be different from the corresponding component of y due to the sorting of y components. It would follow that $\min_{x \neq y} \leq a \leq 3 = \min_{y \neq x}$ in contrary to $\min_{x \neq y} > \min_{y \neq x}$.

Now we need to consider the first appearance of an index j where $z_j < \min_{y \neq z}$. Its existence follows from $y \geq_{mmf} z$ and $z \neq y$. There are three cases: $j > i, j = i$ and $j < i$. In all cases we assume the opposite to the claim to be true, i.e. $\min_{z \neq x} \geq \min_{x \neq z}$ and conclude with a contradiction.

Case 1: $j > i$. Then, the prototype scheme looks like

index:				i		j		
x:	1_+	2_+	3_+	$> 3_+$	$\ldots$			
y:	1_+	2_+	3_+	3_+	4_+	5	6	$\ldots$
z:	1_+	2_+	3_+	3_+	4_+	< 5	$*$	$\ldots$

which means, according to $y \geq_{mmf} z$ that for indices smaller j $y_j = z_j$. Then it follows that $z_i = 3 \neq x_i > 3$ and therefore $\min_{z \neq x} \leq 3$. Since we have assumed that $\min_{z \neq x} \geq \min_{x \neq z}$ then also $\min x \neq z \leq 3$. But only components of x with an index k smaller i can have a value smaller or equal 3 (in order to be $\min_{x \neq z}$), and for all these components $x_k = z_k$.

Case 2: $j = i$. The prototype scheme looks like

index:				i		
x:	1_+	2_+	3_+	> 3	$> 3+$	$\ldots$
y:	1_+	2_+	3_+	3	$3+$	$\ldots$
z:	1_+	2_+	3_+	< 3	$*$	$\ldots$

and similar to case 1 we have $z_i \neq x_i$ thus $\min_{x \neq z} \leq \min_{z \neq x} < 3$, raising the same contradiction since only components with $x_k = z_k$ can have a value $x_k < 3$.

Case 3: $j < i$. The prototype scheme is now

index:			j		i		
x:	1_+	2_+	3	3_+	> 3	$> 3+$	$\dots$
y:	1_+	2_+	3	3_+	3	3_+	$\dots$
z:	1_+	2_+	< 3	$*$	$*$	$*$	$\dots$

In this scheme we have also resorted such that for $k < j$ $x_k = y_k = z_k < 3$. We see that for index j z and x have differing components, and therefore $\min_{z \neq x} < 3$. By assumption then also $\min_{x \neq z} < 3$. But only for $k < j$ there can be $x_k < 3$ in contradiction to $x_k = z_k$ for $k < j$.

In all three cases the assumption $\min_{x \neq z} \leq \min_{z \neq x}$ leaded to a contradiction. Thus $\min_{x \neq z} > \min_{z \neq x}$. $\square$

Using both lemmata, the following can be easily shown.

Theorem 2 *Maxmin fairness over any subset of R_n^+ is a cycle-free relation.*

P: Assume we have a sequence of k vectors with $x_1 \geq_{mmf} x_2 \geq_{mmf} \cdots \geq_{mmf} x_k$. W.l.o.g. we can assume that in this sequence, no two consecutive vectors are equal. Then from Lemma 1 we know that $min_{x_1 \neq x_2} > \min_{x_2 \neq x_1}$. Since $x_2 \geq_{mmf} x_3$ and $x_2 \neq x_3$ by Lemma 2 follows that $\min_{x_1 \neq x_3} > \min_{x_3 \neq x_1}$. Repeated application of the same transition then gives $\min_{x_1 \neq x_k} > \min_{x_k \neq x_1}$. But now, if $x_k >_{mmf} x_1$ this would imply $\min_{x_k \neq x_1} > \min_{x_1 \neq x_k}$ which is not possible. $\square$