Unsorting the Proportional Fairness Relation

Mario Köppen, Kaori Yoshida and Masato Tsuru
Kyushu Institute of Technology
680-4 Kawazu, Iizuka, Fukuoka 820-8502, Japan
Email: mkoeppen@ieee.org, kaori@ai.kyutech.ac.jp, tsuru@cse.kyutech.ac.jp

***Abstract*—Typical problems related to the application of proportional fairness are sparsity of the relation with increasing dimension, and the operator confusion problem. Here, we propose a new fairness relation derived from proportional fairness to handle these problems. The design principle behind this relation is *relational unsorting*: if there is a relation $x(R)y$ between elements x and y from n-dimensional Euclidian space, the unsorted relation $x(uR)y$ holds whenever there is a permutation x^* of the elements of x for which $x^*(R)y$ holds. We apply this concept to proportional fairness, study the properties of the new relation, contrast with another relation based on over-sorting proportional fairness, and provide simulations to demonstrate the ease of ordered proportional fairness for meta-heuristic search.**

***Keywords*-fairness; proportional fairness; ordered proportional fairness; preference modelling**

I. Introduction

Proportional fairness has been shown to maximize the total utility of rate control for elastic traffic in a resource sharing communication network [1][2]. The proportional fair state is characterized as a state vector x of n positive-valued traffic rates such that for any other state $y \in R_+^n$ the inequality

$$\sum_{i=1}^{n} \frac{y_i - x_i}{x_i} \leq 0 \tag{1}$$

holds. However, the total utility does not take an *operator confusion problem* into account. If the operator "forgets" the assignment of the traffic indices i to the real users, the utility of different assignments might not be justified anymore. Taking a very simple example: if for two users the assignment $x = (0.3, 0.2)$ appears to be proportional fair against the assignment $y = (0.3, 0.18)$ (since $0.18 < 0.2$), and the operator forgets who was user 1 and who was user 2 afterwards, then for the other possible assignment $x = (0.2, 0.3)$, proportional fairness does not hold ($0.1/0.2 + (-0.12)/0.3 = 0.1 > 0$). Related issues of global optimization and fairness, while taking user aspects into account, have already been discussed in [3][4]. To handle these issues, we are considering a replacement of the index order of users by ranked order according to the assigned rates. Here, we propose an *ordered proportional fairness* relation and explore the mathematical implications of this formal change, as well as implications for the structure of related optimization. As a result, we will show that within the framework of ordering theory, we can identify a general operation between relations, for which the transiton from proportional fairness to ordered proportional fairness appears to be a special case. Moreover, it is demonstrated that this relation assists meta-heuristic approaches to find its efficient elements, and thus allows for a flexible design of corresponding search algorithms to widen the field of practical applications.

The following Section II will provide the definitions of the proposed relations, and discuss a number of basic properties. Then, in Section III, Monte Carlo simulation results for the proposed relation will be presented. The paper concludes with a discussion in Section IV.

II. Ordered and Anti-Ordered Proportional Fairness

A. Definitions

We represent proportional fairness as a *proportional fair dominance relation* defined as follows:

Definition 1. *A point $x \in R_+^n$ is proportional fair dominating another point $y \in R_+^n$, written as $x >_{pf} y$, if and only if*

$$\sum_{i=1}^{n} \frac{y_i}{x_i} \leq n \tag{2}$$

Then, the best state of this relation, i.e. the state x such that $x >_{pf} y$ for any other y of the feasible space, is generally considered as the proportional fairness state.

We will study a modification of the proportional fairness relation, where the elements of x and y are *sorted* before the condition given by Eq. (1) (also called *indicator expression* in the following) is tested. In the following, a subscript $a_{(i)}$ indicates the i-th largest element of a set A of real numbers a_i.

Definition 2. *A point $x \in R_+^n$ is ordered proportional fair dominating another point $y \in R_+^n$, written as $x >_{opf} y$, if and only if*

$$\sum_{i=1}^{n} \frac{y_{(i)}}{x_{(i)}} \leq n \tag{3}$$

For completeness, we also consider a dual relation:

Definition 3. *A point $x \in R_+^n$ is anti-ordered proportional fair dominating another point $y \in R_+^n$, written as $x >_{aopf} y$, if and only if*

$$\sum_{i=1}^{n} \frac{y_{(i)}}{x_{(n-i+1)}} \leq n \tag{4}$$

B. Basic properties

At first, we will clarify the relations between proportional fairness, ordered proportional fairness, and anti-ordered proportional fairness. This can be summarized in the following theorem.

Theorem 1. *For any x and y from R_+^n, $x >_{pf} y$ implies $x >_{opf} y$ and $x >_{aopf} y$ implies $x >_{pf} y$.*

Proof: For seeing this, we need to interpret the fairness conditions as OWA operators. For definition and basic properties, see the Appendix. In fact, when keeping x fixed, the condition for ordered proportional fairness can be also written as

$$OOWA_w(y) \leq n \tag{5}$$

where the weight vector w has components $w_i = 1/x_{(i)}$, and therefore are non-decreasing. The condition for proportional fairness can be rewritten as

$$OWA_{w*}(y) \leq n \tag{6}$$

where the weight vector $w*$ has components $w*_i = 1/x_i$ and is a permutation of w. So, by Lemma 1 (Appendix), the value of the indicator expression for checking ordered proportional fairness is always smaller than or equal to the value of the indicator expression for proportional fairness. If $OWA_{w*}(y) \leq n$, then also $OOWA_w(y) \leq n$ must hold, and it follows that proportional fairness implies ordered proportional fairness. In a similar way, by using the Anti-Ordered Weighted Averaging (see Appendix), it can be also seen that anti-ordered proportional fairness implies proportional fairness. ■

The theorem can also be summarized by the inequality

$$\sum_{i=1}^{n} \frac{y_{(i)}}{x_{(i)}} \leq \sum_{i=1}^{n} \frac{y_i}{x_i} \leq \sum_{i=1}^{n} \frac{y_{(i)}}{x_{(n-i+1)}} \tag{7}$$

which indicates that proportional fairness is "bounded" by the two new proportional fairness relations.

These relations can be used to better understand the part of space dominated by a point. We recall that the part of R_+^n that is proportional fair dominated by a point x^* can be seen as the part of R_+^n below the tangential hyperplane on the hypersurface given by $\prod x_i = \prod x_i^*$. Figures 1 and 2 show the areas dominated by the point $(4, 1)$ in the case $n = 2$.

In this example, it can be seen that a point y is ordered proportional fair dominated by $(4, 1)$ iff $(4, 1) >_{pf} y$ or $(1, 4) >_{pf} y$, and y is anti-ordered proportional fair dominated by $(1, 4)$ iff $(4, 1) >_{pf} y$ and $(4, 1) >_{pf} y$[1]. From Theorem 1 it can be understood that this is a general characterization of the dominated space.

For given points x and y from R_+^n, we fix y and consider the set X_p of all points that are generated by permutations of the elements of x. Then, if for at least one $x* \in X_p$ the relation $x* >_{pf} y$ holds, by Theorem 1 $x* >_{opf} y$ follows,

[1]Note that the blue dashed line shows the curve $x \cdot y = 1 \cdot 4$, and the dominated parts are bounded by tangents to this curve

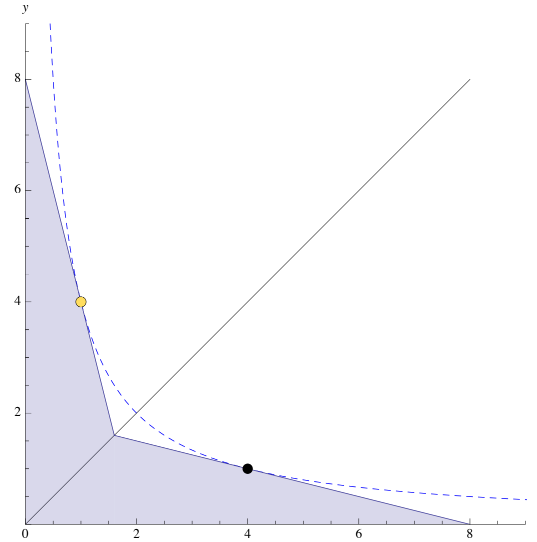

Fig. 1. Set of points *ordered proportional fair dominated* by the point $(4, 1)$. It can be seen as the set of points proportional fair dominated by the point $(4, 1)$ (black dot in the figure) **or** proportional fair dominated by its mirror point $(1, 4)$ (yellow dot in the figure).

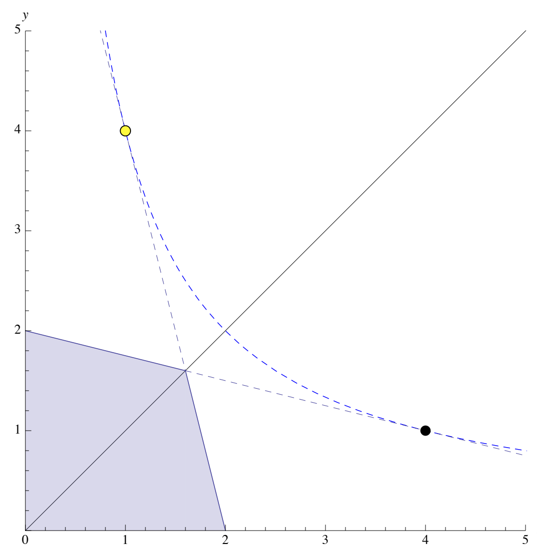

Fig. 2. Set of points *antiordered proportional fair dominated* by the point $(4, 1)$. It can be seen as the set of points proportional fair dominated by the point $(4, 1)$ (black dot in the figure) **and** proportional fair dominated by its mirror point $(1, 4)$ (yellow dot in the figure).

and then $x >_{opf} y$, since x has the same components like $x*$, only sorted in non-increasing order. Therefore, the value of the indicator expression for ordered proportional fairness does not change. On the other hand, if $x >_{opf} y$ while the components of x are not sorted, must correspond to $x* >_{pf} y$ for the $x* \in X_p$ where for all i $x*_i = y_{(i)}$. In summary, the fact that at least one element of X_p proportional fair dominates y is a necessary and sufficient condition for $x >_{opf} y$.

In a similar way, if there is any $x* \in X_p$ where $x*$ is not proportional fair dominating y, then $x* >_{aopf} y$ is also not possible, as otherwise, $x* >_{pf} y$ would follow. But $x* >_{aopf}$

y and $x >_{aopf} y$ mean the same thing, because $x*$ is just a permutation of x. On the other hand, there must be one sorting $x*$ of the components of x that corresponds to the inverse sorting of the components of y, and for this, $x* >_{aopf} y$ and $x* >_{pf} y$ are formally the same. Thus we can also see that it is necessary and sufficient for x anti-ordered proportional fair dominating y that all elements of X_p are proportional fair dominating y.

We remark that all three relations are not complete, i.e. there are pairs (x, y) where neither $x >_r y$ nor $y >_r x$ holds. However, as will be demonstrated below, ordered proportional fairness can be seen as a "nearly complete" relation.

Like proportional fairness, also ordered proportional fairness and anti-ordered proportional fairness are not transitive. It suffices to provide counterexamples: for ordered proportional fairness, consider the three points $x = (47, 43), y = (36, 53)$ and $z = (5, 91)$. Then we have for the values of the indicator functions $I_{xy} = 53/47 + 36/43 = 1.96 \leq 2, I_{yz} = 1.86 \leq 2$ but $I_{xz} = 2.05 > 2$. So this is a case where $x >_{opf} y$ and $y >_{opf} z$ but not $x >_{opf} z$.

For anti-ordered proportional fairness, we can consider $x = (63, 36)$, $y = (41, 47)$ and $z = (9, 68)$. Then $I_{xy} = 47/36 + 41/63 = 1.96 \leq 2$ and $I_{yz} = 1.85 \leq 2$ but $I_{xz} = 2.03 > 2$, thus $x >_{aopf} y$ and $y >_{aopf} z$ but not $x >_{aopf} z$.

However, both relations are not cyclic (also like proportional fairness), which means that there is no sequence of m points x_i such that $x_i >_{opf|aopf} x_{i+1}$ for $i = 1, \ldots, (m-1)$ and $x_m >_{opf|aopf} x_1$. For proportional fairness, this can be verified by the fact that $x >_{pf} y$ implies that x has also a larger product of components[2], $\prod_i x_i > \prod_i y_i$ and thus a point with a smaller product of components than $\prod_i x_i$ can never dominate x. This implies that for $x >_{pf} y$ and $y >_{pf} z$ the products of components of z is smaller than $\prod_i x_i$, and z cannot dominate x. The same fact can be seen for any lengths of transitivity sequences in a similar manner.

The fact that also anti-ordered proportional fairness is cycle-free follows directly from Theorem 1. If there would be a cycle for the anti-ordered proportional fair dominance relation, this would imply the same cycle for the proportional fair dominance relation, which cannot exist.

For the case of ordered proportional fairness, it needs to consider the specific sorting of the components in the indicator expressions. Since in all cases the components are ordered in a non-increasing manner, the singular terms in the indicator expressions are all ordered in the same way. It means if we compare x and y we have an order for the terms $y_{(i)}/x_{(i)}$ which is the same order as for the terms $z_{(i)}/y_{(i)}$ in the comparison of y and z and $z_{(i)}/x_{(i)}$ in the comparison of x to z. Thus, the indicator expressions are formally identical to the expression for proportional fairness, just with all components of all points sorted by size. Then also here, cycle-freeness of proportional fairness implies cycle-freeness of the ordered proportional fairness.

[2]Note that two different points with the same product of components can never dominate each other.

C. Links to other relations

As a preparation, we provide a number of related definitions.

Definition 4. *For any x and y from R^n it is said that x (strictly) Pareto dominates y, written as $x >_p y$, if and only if*

$$\forall_i x_i \geq y_i \wedge \exists_j x_j > y_j \tag{8}$$

(note that there can be a corresponding definition using $<$ and $\leq$). A stronger version of this definition:

Definition 5. *For any x and y from R^n it is said that x totally Pareto dominates y, written as $x >_{tp} y$, if and only if*

$$\min[x_i] > \max[y_i] \tag{9}$$

We also consider the definition for lexicographic minimum relation.

Definition 6. *For any x and y from R^n it is said that x lexmin dominates y, written as $x >_{lm} y$, if and only if for the largest i such that $x_{(i)} \neq y_{(i)}$ the inequality $x_{(i)} > y_{(i)}$ holds.*

Last but not least, we consider two extensions of the ordered and anti-ordered proportional fairness, in analogy to the corresponding extension of proportional fairness to α-fairness:

Definition 7. *A point $x \in R^n_+$ is ordered α-fair dominating another point $y \in R^n_+$, written as $x >_{o\alpha} y$, if and only if*

$$\sum_{i=1}^{n} \frac{y_{(i)} - x_{(i)}}{x^{\alpha}_{(i)}} \leq 0 \tag{10}$$

Definition 8. *A point $x \in R^n_+$ is anti-ordered α-fair dominating another point $y \in R^n_+$, written as $x >_{ao\alpha} y$, if and only if*

$$\sum_{i=1}^{n} \frac{y_{(i)} - x_{(n-i+1)}}{x^{\alpha}_{(n-i+1)}} \leq 0 \tag{11}$$

Now we can make some additional statements:

1) If the components of x are stronger differing, the ordered proportional fair dominance relation resembles the lexmin relation, while the anti-ordered proportional fair dominance relation resembles the total Pareto dominance relation.
2) If the components of x are becoming more similar, both, the ordered proportional fair dominance relation and the anti-ordered proportional fair dominance relation resemble the proportional fair dominance relation.
3) Pareto dominance implies ordered proportional fair dominance, since Pareto dominance implies proportional fair dominance. However, there is no relation between Pareto-dominance and anti-ordered proportional fair dominance. Only total Pareto dominance implies anti-ordered proportional fair dominance.
4) Since the space that is ordered proportional fair dominated by a point x is usually not convex, even in case of convex feasible subspaces of R^n_+ the maximum set (i.e. set of non-dominated points) of this relation can

have more than one element (in contrary to proportional fairness, maxmin fairness etc.).

III. Monte Carlo Simulations

In addition to basic mathematical properties, the important additional practical application aspect is the tractability of the accompanying search problem: given a feasible set of vectors from a specific problem domain, how can we find the maximum sets (also called set of maximal, efficient, or non-dominated elements) for a given relation? In a recent work [5], we have already investigated the options to use meta-heuristic search algorithms, and demonstrated their efficiency. But the study also illustrated the important aspects of the relations itself, in order to pose a tractable search problem, and with regard to the chance of random occurrence of a relation between two random points. It can be easily seen that a sparse relation inhibits the initial explorative stage of such algorithms. Therefore, we want to focus on this aspect when evaluating the newly proposed relations, and provide a number of related Monte Carlo simulations to estimate corresponding probabilities and probability distributions.

A. Probability of occurrence

In this part, we want to study the probability of occurrence of relations between random vectors with increasing dimension n. It is known to fall exponentially for the Pareto dominance relation, and can be expected to be 0.5 for complete relations like lexmin. Other estimates might be harder to find, so we sampled 100,000 pairs x and y of random points from $(0,1)^n$ and counted the number of occurrences of the relation $x >_R y$ for various relations. Table I gives an overview of the results for dimensions up to 100.

The exponential decay of the Pareto dominance relation $>_p$ can be confirmed, for dimensions 15 onwards it is virtually not present anymore (a large hindrance for meta-heuristic approaches to multi-objective optimization with larger number of objectives). Proportional fairness $>_{pf}$ also decreases, but it might be a conjecture if this refers to an exponential decay as well. As can be expected from its relation to total Pareto dominance, which is even stronger than Pareto dominance, also anti-ordered proportional fairness decays strongly with increasing dimension, but nevertheless slower than Pareto dominance. The maxmin fairness falls about linearily with the dimension, which can be also rather easily estimated. Then, ordered proportional fairness stays nearly constant at a value around 40%. This means that ordered proportional fairness is close to a complete relation having 50%, and it is even possible that the probability is converging (at least, it seems to decay very slowly). This can be related to the fact that this relation has a much larger dominated space by a point x than for example proportional fairness. In fact, it is fusing an exponentially increasing number of proportionally fair dominated spaces (one for each permutation of the components of x), and this can "compete" with the exponential decrease of the volume of spaces proportionally fair dominated by a point with increasing dimension.

TABLE I
Frequency of occurrence of various relations with increasing dimension n among 100,000 random samples from $(0,1)^n$.

n	$>_p$	$>_{pf}$	$>_{opf}$	$>_{aopf}$	$>_{mmf}$
1	50021	50021	50021	50021	50021
2	25032	40344	46784	33868	33587
3	12523	33203	45101	23520	25044
4	6159	27852	43984	16826	19433
5	3067	23305	43117	11920	16675
6	1404	20300	43155	8521	14133
7	652	17266	42304	6237	12682
8	434	14960	41943	4693	10901
9	282	13209	41794	3530	10374
10	117	11219	41512	2614	8527
11	25	9705	41575	2057	7697
12	39	9029	41217	1558	7329
13	17	7461	41198	1219	7698
14	24	6704	41252	931	7905
15	11	5769	41119	743	5766
16	2	5001	40845	581	6416
17	0	4494	40785	477	5534
18	0	3792	40508	406	5552
19	0	3586	40758	337	5876
20	0	3103	40853	274	5092
25	0	1630	40865	122	4108
30	0	994	40628	51	2916
35	0	739	40704	18	2793
40	0	510	40766	6	2475
45	0	344	40858	1	2102
50	0	217	40744	2	2274
60	0	91	40566	0	1729
70	0	23	40735	0	2208
80	0	6	41165	0	862
90	0	40	41779	0	1422
100	0	0	41720	0	687

We also studied conditional properties, to indicate the relation between ordered proportional fairness, the maximization of product of components, and the maximization of the smallest component. The reason is that in case of convex feasible spaces, the proportional fairness has a best element, which is also maximizing the product of its components. This cannot be expected from the ordered proportional fairness as well, as the subspace dominated by a point is not always convex, but a relation to component product maximization might still exist.

This is confirmed by the result shown in Fig. 3. There, a number of conditional probabilities $p(>_1 \mid >_2)$ have been sampled from random vectors from $(0,1)^n$ (100,000 samples in each case). The used relations were ordered proportional fairness $>_{opf}$, lexmin $>_{lm}$ and a further relation $>_{prod}$ which holds between x and y iff $\prod_i x_i > \prod_i y_i$. The case $p(>_{prod} \mid >_{opf})$ is not shown, since ordered proportional fairness always implies a larger product of components (this is the case for proportional fairness, and ordered proportional fairness always corresponds to a particular case of proportional

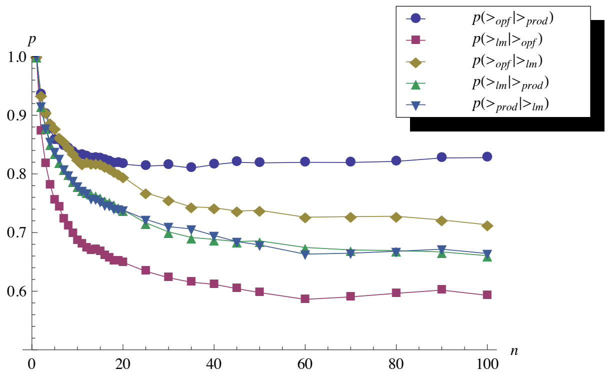

Fig. 3. Sampled conditional probabilities that a random vector x from $(0,1)^n$ $>_1$-dominates another random vector y given that $x >_2 y$, where the relations $>_1$ and $>_2$ are among the relation of having larger product of components($>_{prod}$), lexmin, and ordered proportional fairness (100,000 samples for each dimension n). The case $p(>_{prod} \mid >_{opf})$ is not shown, since this is always 1.

fairness). So, this value is always 1.

Two things can be seen in the result: the first is that the probabilities seem to converge, or at least decay very slowly with increasing dimension. Which of these is the case might be subject of further investigation (as it includes the rather fundamental question if the chance that a larger product of components implies a larger minimum of the components and *vice versa* is converging to a value around 0.65 for large dimensions). We can also see that the "mutual information" between ordered proportional fairness and component product maximization is rather high, highest in the cases shown here. It also confirms that the mutual information between ordered proportional fairness occurrence and lexmin is comparable to the component product maximization, while the values itself are differing (for large n, the chance that from $x >_{opf} y$ also $\min_i[x_i] > \min_i[y_i]$ follows is about 60%, which is smaller than the same chance for having a larger product of components, while the chance that from a larger minimum of x compared to y also $x >_{opf} y$ follows is higher than for the product, with a value of about 75%).

In summary, we see that ordered proportional fairness is rather independent from proportional fairness, but stronger related to maximization of the product of components.

B. Distribution of fairness indicator values

The following experiment was performed: 10000 times pairs of random vectors x and y with $n = 50$ components each from $(0,1)$ were generated, and the ordered proportional fairness, anti-ordered proportional fairness, and proportional fairness indicator values were computed (i.e. the values that are compared with the dimension in the definition of the relations). Then, 20 bins were defined for the frequency of values of the proportional fairness indicator between the ordered proportional fairness indicator and the anti-ordered proportional fairness indicator.

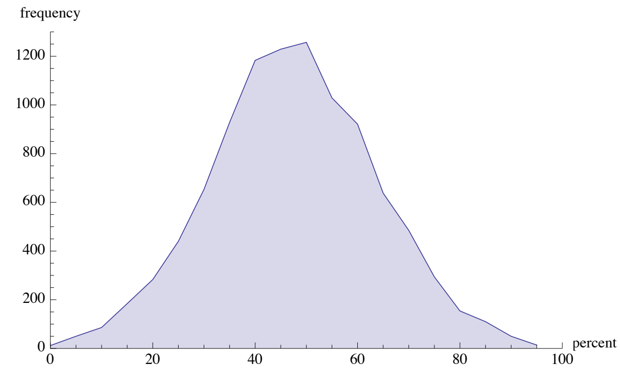

Fig. 4. Distribution of proportional fairness indicator values between ordered proportional fairness and anti-ordered proportional fairness indicator values for 10000 random samples of dimension 50.

Figure 4 shows that the distribution resembles a normal distribution.

IV. Discussion

In comparison, ordered proportional fair dominance relation seems to be the more interesting relation, compared to anti-ordered proportional fair dominance relation, for several reasons: it can mediate between proportional fairness for cases, where components of x become more similar, and the lexmin dominance relation, where one component becomes much smaller than the other. Also, since the part of the space dominated by x is (much) larger than the part of the space proportional fair dominated by x, meta-heuristic approaches can find the maximum set elements of this relation (much) more easily. If the main concern is still about proportional fairness: also here, ordered proportional fairness is helpful, since the maximum (efficient, non-dominated) set of ordered proportional fairness is a subset of the maximum set for proportional fairness, so at least a few proportional fair states can be found, and then more easily.

The given arguments also show that ordered proportional fairness can be seen as extension of proportional fairness "up to a permutation". This is formally the same "transformation" of a relation that leads from maxmin fairness to lexmin fairness. Thus, we have also achieved a formal way to expand any relation r as a subset of the direct product of two sets A^n and B by a procedure of un-sorting: $x >_{u(r)} y$ holds iff there is at least one permutation of the elements of x such that $x >_r y$ holds. We can do similarly for the processing of over-sorting and requiring $x >_r y$ for *all* permutations. Applied to other relations, over-sorting applied to Pareto-dominance, as well as maxmin fairness, gives total Pareto dominance, and un-sorting gives a relation that has up to our knowledge not been studied so far, based on the comparison $x_{(i)} \geq y_{(i)}$. We can go the other way as well, and seek for known relations r different de-sorting relations $d(r)$ such that un-sorting $d(r)$ gives r. Without giving much details here, in at least one case, the exponential OOWA [6] can serve as r to yield a new relation that has some resemblance to a bounded version of proportional fairness. In summary, we have to acknowledge the fact that fairness relations are probably not sparse but establish a rich category with numerous mutual relationships.

References

[1] F. Kelly, "Charging and rate control for elastic traffic," *Eur. Trans. Telecomm.*, vol. 8, pp. 33–37, Jan./Feb. 1997.

[2] D. Bertsekas and R. Gallager, *Data Networks*. Englewood Cliffs, N: Prentice Hall, 1992.

[3] M. Köppen, K. Yoshida, M. Tsuru, and Y. Oie, "Fairness-based global optimization of user-centric networks," in *INC, IMS and IDC, 2009. NCM '09. Proc. Fifth International Joint Conference on*, Seoul, Korea, Aug. 2009, pp. 441–443.

[4] J. Y. L. Boudec. Rate adaptation, congestion control and fairness: a tutorial. [Online]. Available: http://ica1www.epfl.ch/PS_files/LEB3132.pdf

[5] M. Köppen, R. Verschae, K. Yoshida, and M. Tsuru, "Comparison of evolutionary multi-objective optimization algorithms for the utilization of fairness in network control," in *Proc. 2010 IEEE International Conference on Systems, Man, and Cybernetics (SMC 2010), Istanbul, Turkey*, 2010, pp. 2647 –2655.

[6] M. Köppen, J. Okamoto, and A. Honda, "Fuzzy fusion fairness relations for the evaluation of user preference," in *2011 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2011), Proceedings*, Taipei, Taiwan, June 2011.

Appendix

Proof for minimizing property of the Ordered Ordered Weighted Averaging

We recall the definition of the Ordered Weighted Averaging operator.

Definition 9. *Given a point x from R_n and a set of weights $w \in R_n$, the Ordered Weighted Averaging (OWA) of x by w is defined as*

$$OWA_w(x) = \sum_{i=1}^{n} w_i x_{(i)} \tag{12}$$

where $x_{(i)}$ denotes the i-th largest component of x.

The Ordered-Ordered Weighted Averaging operator is a special case of the OWA operator, where additionally the weights are sorted in non-decreasing order, i.e. in the corresponding expression, the largest $x_{(1)}$ is multiplied with the smallest weight etc.

Definition 10. *Given a point x from R_n and a set of weights $w \in R_n$, where the weights are sorted in non-decreasing order (i.e. $w_1 \leq w_2 \leq \cdots \leq w_m$), the Ordered-Ordered Weighted Averaging (OOWA) of x by w is defined as*

$$OOWA_w(x) = \sum_{i=1}^{n} w_i x_{(i)} \tag{13}$$

where $x_{(i)}$ denotes the i-th largest component of x.

For this operator, we will show:

Lemma 1. *Given a set of n weights w in non-decreasing order, and any permutation $w*$ of these weights. Then for any $x \in R^n$*

$$OOWA_w(x) \leq OWA_{w*}(x) \tag{14}$$

Proof: We will use bracketed subscripts to refer to the different orderings. By $x_{(i)}$ we indicate the i-th largest element of x, by $w_{(i*)}$ the i-th element in the permutation $w*$ of the weights. In this notation

$$OWA_{w*}(x) = \sum_{i=1}^{n} w_{(i*)} x_{(i)} \tag{15}$$

and

$$OOWA_w(x) = \sum_{i=1}^{n} w_i x_{(i)} \tag{16}$$

We also define

$$h_i = x_{(n-i+1)} - x_{(n-i+2)} \tag{17}$$

for $i > 1$ and $h_1 = x_{(n)}$. Then

$$x_{(i)} = \sum_{k=1}^{n-i+1} h_k \tag{18}$$

and

$$\begin{aligned} OWA_{w*}(x) &= \sum_{i=1}^{n} \left[w_{(i*)} \sum_{k=1}^{n-i+1} h_k \right] \\ &= \sum_{i=1}^{n} \left[h_i \sum_{k=1}^{n-i+1} w_{(k*)} \right] \\ &\geq \sum_{i=1}^{n} \left[h_i \sum_{k=1}^{n-i+1} w_k \right] \\ &= \sum_{i=1}^{n} \left[w_i \sum_{k=1}^{n-i+1} h_k \right] = OOWA_w(x) \end{aligned}$$

since the weights are sorted in non-decreasing order and thus, the sum of the first k weights will be always smaller or equal to the sum of the first k permuted weights. ■

In a similar way, we can consider an Anti-Ordered Weighted Averaging operator (AOWA) by requiring the weights to be sorted in non-increasing order, and show that for any permutation $w*$ of the weights

$$AOWA_w(x) \geq OWA_{w*}(x) \tag{19}$$

holds.