A Fuzzy Scheme for the Ranking of Multivariate Data and its Application

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Abstract— **This paper presents a generic fuzzy scheme for the ranking of multivariate data. The scheme is based on a comparison function of two numbers. The comparison function values are fused by a T-norm for all components of two vectors, giving the comparison values. For each vector in a set of vectors, the smallest value of its comparison values with all other elements of the set is assigned to this vector as its ranking value. Then, all further processing is based on the ranking values alone. As a suitable comparison function, bounded division is identified. The application of the scheme to define a color morphology and an evolutionary multiobjective optimization algorithm is demonstrated.**

I. Introduction

In several research fields like multi-objective optimization, color image processing, multi-sensorial fusion, just to name a few, the need often arises to rank multivariate data. In general, there is no unique way to achieve this [1]. The most common approach is to use a scalar function $f(\vec{a})$, mapping a real valued vector $\vec{a}$ into R, and to sort a set of vectors $\vec{a}_i$ with respect to the magnitudes of the function values $f(\vec{a}_i)$. However, it can be shown that not all ranking schemes can be based on this approach. Considering the usage of fuzzy concepts in such a context might help to overcome this limitation. The basic idea is to represent a set of multivariate data by a set of ranking values from [0,1] and to restrict all further ranking operations to this set. The essential extension of the scalar function approach here is that the ranking values can depend on the other elements of the set. These ranking values, in a fuzzy sense, represent the degrees of any data value to be of that rank, which it has in the ranking of its corresponding ranking value among all the ranking values. For example, selecting the element(s), to which the highest ranking value was assigned, can be used to expand the maximum operation to multivariate data.

This approach is considered in more detail in this paper. Section II is considering the fuzzification of the Pareto dominance relation in general, while section III gives a ranking scheme for multivariate data, which is based on the fuzzification of Pareto dominance. Section IV considers two possible applications of this scheme: the definition of color morphology operators (dilation and erosion) and the definition of an evolutionary multiobjective optimization algorithm. The paper finishes with a short conclusion section.

II. Fuzzification of Pareto Dominance Relation

In this section, we will consider approaches to the fuzzification of the Pareto dominance relation between two vectors. For two vectors $\vec{a}$ and $\vec{b}$ it is said that $\vec{a}$ *(Pareto-)dominates* $\vec{b}$, when each component of $\vec{a}$ is less or equal to the corresponding component of $\vec{b}$, and at least one component is smaller:

$$\vec{a} >_D \vec{b} \longleftrightarrow \forall i(a_i \leq b_i) \wedge \exists k(a_k < b_k). \tag{1}$$

Note that in a similar manner Pareto dominance can be related to $>$-relation, depending on the application context.

The subset of all vectors of a set M of vectors, which are not dominated by any other vector of M is the Pareto set (also Pareto front). The Pareto set for univariate data (single objective) contains just the maximum of the data.

There are basically two ways of considering a fuzzy extension of this dominance relation. The first way is to extend the logical relation between two vectors $\vec{a}$ and $\vec{b}$ by a (non-symmetric and monotone) degree function $D(\vec{a},\vec{b})$, which assigns 0 to the vector pair if $\vec{b}$ dominates $\vec{a}$, which assigns 1 if $\vec{a}$ dominates $\vec{b}$, and a value between 0 and 1 if there is no dominance relation between $\vec{a}$ and $\vec{b}$. We assume that all vector components are from $[0,1]$. Figure 1 gives the set-up of such a function for the case of vectors of two components (the rather complex case of vectors of more than two components will not be considered here). If the two vectors are denoted

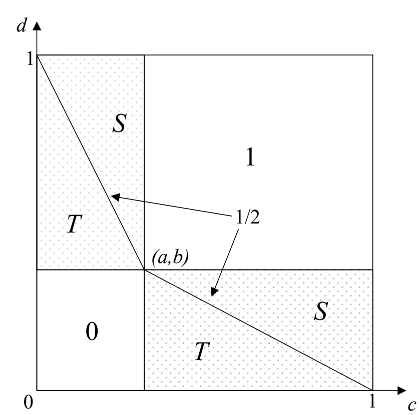

Fig. 1. Specification of a degree function $D((a,b),(c,d))$ by splitting the unit square into four areas. For the two hatched areas, T- and S-norms can be used to define D, as e.g. $T(\frac{c-a}{1-a}, \frac{d}{b})$ for the T-norm expression in the bottom right area.

by (a,b) and (c,d), then vector (a,b) splits the unit square into four areas. It can be seen that the degree function is not continuous at the point $(a,b) = (c,d)$ itself, but it is reasonable here to make a fixed assignment like $D((a,b),(a,b)) = 1/2$. For the bottom left area $c \leq a, d \leq b$ the function D has the unique value 0, and for the top right area $c \geq a, d \geq b$ the unique value 1. For giving an appropriate extension of the degree function for the bottom right and top left area, we may use a monotone function $g(x,y)$ mapping $[0,1] \times [0,1]$ onto $[0,1]$ and fulfilling the following two boundary conditions:

$$\begin{aligned} g(0,y) &= 0 \\ g(x,1) &= 1 \end{aligned} \tag{2}$$

Then, g is mapped from the unit square to the rectangle with corner (a,b) and $(1,0)$ for the bottom right area, and onto the rectangle spanned by the corner points $(0,1)$ and (a,b) and rotated counterclockwise by $\pi/2$ for the top left area. While the first property characterizes g as a T-norm in its first argument, the second property characterizes g as a S-norm in its second argument. Further studies will consider the corresponding class of such functions. Here, we will give only two examples for such functions g:

$$g_1(x,y) = \begin{cases} T(x,y) & : \quad x+y<1 \\ 1/2 & : \quad x+y=1 \\ S(x,y) & : \quad x+y>1 \end{cases} \tag{3}$$

with $T(x,y)$ and $S(x,y)$ a corresponding pair of T-norm and T-conorm (or S-norm), or

$$g_2(x,y) = \frac{x}{\sqrt{x^2+(y-1)^2}} \tag{4}$$

and $g_2(0,1) = 1/2$. The second definition relates the degree function to the sinus of the angle $\angle(0,0)(0,1)(x,y)$.

It has to be noted that this is just the definition for the case of vectors of two components. The extension to vectors of more components is not straightforward and is making use of rather complex functions. Also, considering the degree values within a set of vectors, all dominating and dominated vectors will not be distinguishable since the degree functions uniquely assigns 1 or 0 to them.

Therefore, we propose a second way of fuzzifying the dominance relation, which somehow goes halfway the definition of a degree function that was just given. This is achieved by skipping either the requirement of D to be 0 in case $\vec{a}$ is dominated by $\vec{b}$, or by skipping the requirement $D(\vec{a},\vec{b}) = 1$ in case $\vec{a}$ is dominating $\vec{b}$. Then, vectors dominating other vectors become *comparable* according to their degree of dominating other vectors. How such a degree function can be constructed and used to rank multivariate (vector) data will be given in the next section.

III. Multivariate Fuzzy Ranking Scheme

A generic fuzzy ranking scheme for a set S of multivariate data (vectors) $\vec{a}_i$ with real-valued components a_{ij} and $1 \leq i \leq N$ is presented and studied. The scheme is based on the provision of a *comparison function* $f_x(y) : R \times R \rightarrow [0,1]$ and a T-norm. Then, the following two steps are performed:

1) We compute the *comparison values* for any two vectors $\vec{a}_i = (a_{ik})$ and $\vec{a}_j = (a_{jk})$ by $c_{\vec{a}_i}(\vec{a}_j) = T(f_{a_{ik}}(a_{jk}) | k = 1, \ldots, N)$ with N the number of components of each vector.
2) We compute the *ranking values* for any element $\vec{a}_i$ of S by $r_S(\vec{a}_i) = \max[c_{\vec{a}_i}(\vec{a}_j) | j \neq i]$.

Then, we consider vectors with lower numerical ranking values to be on a higher ranking position. For step 2, instead of max the min operator can be used as well, depending on the ranking to be favoured in increasing or decreasing order.

Of particular interest are ranking values with the following three properties:

- *Scale independency*: if all vector components are multiplied by a scalar k, the ranking values will not change.
- *Relative ranking values*: We consider a ranking between two vectors $\vec{a}$ and $\vec{b}$ as absolute if the ranking values of both vectors in any set will always be in the same ranking relation. If these relation depends on the other elements of the set, we consider the ranking as set-relative.
- *Marginality*: If the data are univariate, the ranking value is lowest for the maximum of the values.

All these properties are fulfilled when the comparison function is a bounded division

$$f_x(y) = \begin{cases} \frac{\min[x,y]}{y} & : \quad y \neq 0 \\ 1 & : \quad y = 0 \end{cases} \tag{5}$$

or

$$f_x(y) = \begin{cases} \frac{\min[x,y]}{x} & : \quad x \neq 0 \\ 1 & : \quad x = 0 \end{cases} \tag{6}$$

and the algebraic (or product) norm is used as T-norm. When we use this function, the comparison values might have different interpretations, depending on the application context. Initially, this function has been introduced by Kosko to compute subsethood degrees of fuzzy sets [7], but it can be considered e.g. a "fuzzy dominance" relation as well. Using bounded division and product norm, the fuzzification of Pareto dominance relation can be written as follows: It is said that vector $\vec{a}$ *dominates* vector $\vec{b}$ by degree μ_a with

$$\mu_a(\vec{a},\vec{b}) = \frac{\prod_i \min(a_i,b_i)}{\prod_i a_i} \tag{7}$$

and that vector $\vec{a}$ *is dominated by* vector $\vec{b}$ at degree μ_p with

$$\mu_p(\vec{a},\vec{b}) = \frac{\prod_i \min(a_i,b_i)}{\prod_i b_i} \tag{8}$$

Note that the definitions differ in the denominator and thus are not symmetric: "dominating by degree μ" and "being dominated by degree μ" have different fuzzy values. For $\vec{a}$ Pareto-dominating $\vec{b}$, $\mu_a(\vec{a},\vec{b}) = 1$ and $\mu_p(\vec{b},\vec{a}) = 1$, but $\mu_p(\vec{a},\vec{b}) < 1$ and $\mu_a(\vec{b},\vec{a}) < 1$. Figure 2 gives a numerical example for the fuzzy Pareto dominance considered here.

This ranking scheme is obviously scale-independent, and it will become the algebraic norm max or min if $N = 1$, thus it is

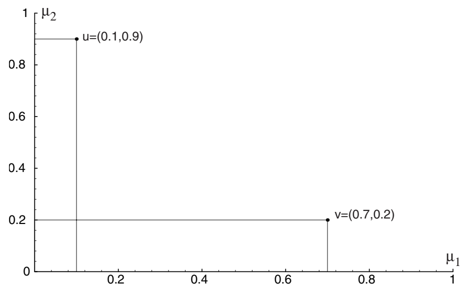

Fig. 2. Definition of Fuzzy-Pareto-Dominance. Here, u dominates v by degree $0.1 \cdot 0.2/0.1 \cdot 0.9 = 0.\bar{2}$ and is dominated by v by degree $0.1 \cdot 0.2/0.7 \cdot 0.2 \approx 0.143$.

a marginal operation. It also fulfills the requirement for being a set relative ranking. It can be shown that there is no scalar function of vector components of one vector at all, which will give the same ranking of the vectors of a set M. This can be shown by a simple counterexample.

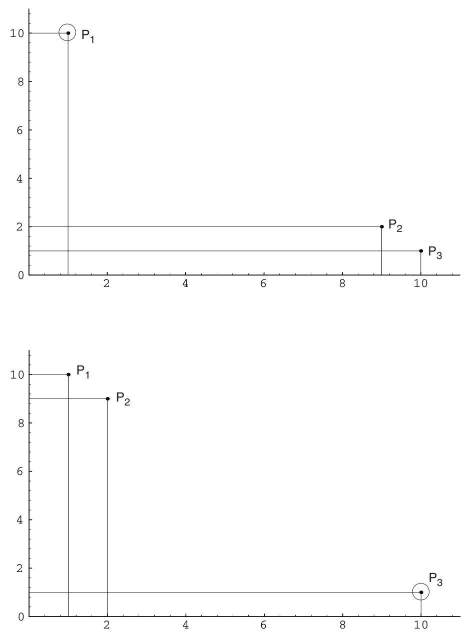

Fig. 3. Counterexample: no scalar function of P components can give the same ranking as the proposed FPD ranking.

Consider figure 3. We assume, that there is a scalar function f, which gives the same ranking as the fuzzy ranking scheme. If we take the set of three vectors $\{(1,10),(9,2),(10,1)\}$, the vector with lowest ranking value is (1,10). If we take the set $\{(1,10),(2,9),(10,1)\}$, the vector with lowest ranking value will be (10,1). If there would be such an f, it has to be $f(1,10) > f(10,1)$ from the first case, but also $f(1,10) < f(10,1)$ from the second case. This is a contradiction, hence there is not such a f. The ranking value of P_2 strictly depends on the set of points, to which the vector belongs.

IV. Applications

A. Color Morphology

In color image processing, the ranking scheme can be used to extend the definition of dilation and erosion to color images, thus it bases other operations of color mathematical morphology as well [2][3][10][5]. A color image is usually given by a mapping I of a set $B = \{0,\ldots,w-1\} \times \{0,\ldots,h-1\}$ into a set of feasible color values C. The color values are related to a color space. Here, we are considering the intensity-based technical color space RGB, with each color component red (R), green (G) and blue (B) taken from the set $\{0,\ldots,c_{max}\}$ (usually $c_{max} = 255$). The mapping can be written as $I(x,y) = (r_{(x,y)}, g_{(x,y)}, b_{(x,y)})$ where $(x,y) \in B$ is the so-called image coordinate, and the image function I assigns a three-valued vector to each image coordinate (the pixel).

In the image processing discipline of Mathematical Morphology, this definition is extended by the so-called *structuring element* (SE) (see [8] and [9] for an comprehensive introduction into mathematical morphology). The SE is given by a *neighborhood operator* that assigns a set of image coordinates to any pixel. Commonly used neighborhoods are the four (or eight) direct neighbors of a pixel in the image domain. Then, the goal is to define an operation that selects one out of the neighboring pixels and replaces the color value of the pixel itself with the color value of the selected pixel from the SE. Thus, a color image I_1 is transformed into a color image I_2. If the selection from SE is based on some ranking concept, the corresponding operation is called a dilation or erosion. There are different proposals for extending dilation and erosion for grayscale images to color images.

The proposed fuzzy ranking scheme can be used to define various morphological operations, different from the ones proposed so far. We have to distinguish between the two choices for the bounded division and the ranking order, thus yielding four different operations. For the operations described in Table I, $a = (a_i)$ and $b = (b_i)$ refer to two color vectors that have to be compared ($i \in (r,g,b)$) and S to the set of neighbor pixels of a. The terms "active" and "passive" in the names of the operations should indicate that either the degree of a dominating b (*active*) or the degree of a being dominated by b (*passive*) is considered.

TABLE I

Definitions for color morphologies based on the proposed fuzzy ranking scheme.

Operation	$f_{a_i}(b_i)$	$c_a(b)$	$r_S(a)$	selection
active erosion	$\frac{\max[a_i,b_i]}{a_i}$	$\prod_i f_{a_i}(b_i)$	$\min_{S\backslash a}[c_a(b)]$	argmax
active dilation	$\frac{\min[a_i,b_i]}{a_i}$	$\prod_i f_{a_i}(b_i)$	$\max_{S\backslash a}[c_a(b)]$	argmin
passive erosion	$\frac{\min[a_i,b_i]}{b_i}$	$\prod_i f_{a_i}(b_i)$	$\max_{S\backslash a}[c_a(b)]$	argmin
passive dilation	$\frac{\max[a_i,b_i]}{b_i}$	$\prod_i f_{a_i}(b_i)$	$\min_{S\backslash a}[c_a(b)]$	argmax

In Table I it can be seen how different these operations behave, as e.g. in the dominance case. To consider the active dilation as an example: in case a is (max)dominated by any b from its neighborhood, a will never be selected. If all components of a are smaller or equal than the components of b, $a_i \leq b_i$, then for all i $f_{a_i}(b_i) = 1$, giving $c_a(b) = 1$ and a has the ranking value $r_S(a) = 1$. This is the highest possible value for a ranking value, so the selection by argmin will never select a (at least b has a ranking value below 1). The same can be seen for the other three definitions. Here is, in summary, the sequence of steps that are performed for any image coordinates (x,y) in order to apply any of these four operations:

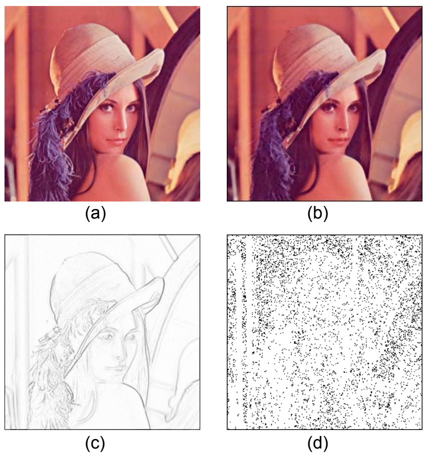

Fig. 4. Application of active erosion onto color Lena image: (a) original image; (b) result of active erosion applied with nearest neighbors as SE; (c) difference between (a) and (b); (d) positions, for which active erosions selects different than the argmin of the sum of color values $r+g+b$.

1) Select all neighbours of (x,y) by the SE. The RGB color values of all the neighbours comprise the set S.
2) For each $a \in S$ compute all comparison values $c_a(b)$ for any $b \in S \setminus a$.
3) From the set of all comparison values for a derive the ranking value $r_S(a)$ of a.
4) Select the a from S according to the selection criteria (argmin or argmax of the ranking values). In case there are more than one a with the same maximum (or minimum) ranking value, select by an additional criteria (as the highest/lowest sum of components).
5) Assign the color value of the selected pixel at position (x,y) in the result image.

As an additional operation mode, the subset of all dominated vectors can be removed in advance. Figure 4 shows the result of applying active erosion on the standard Lena image, and compares the result to the original and the scalar-function based ranking (i.e. from the neighborhood the color value with the smallest sum of r,g and b is selected, see introduction section).

B. Evolutionary Multiobjective Optimization

In this subsection, we will show how the fuzzy ranking scheme easily extends a standard genetic algorithm to the multi-objective case.

In multiobjective optimization, the optimization goal is given by more than one objective to be extreme. Formally, given a domain as subset of R^n, there are assigned m functions $f_1(x_1,\ldots,x_n),\ldots,f_m(x_1,\ldots,x_n)$. Usually, to solve an multiobjective optimization problem, the Pareto set of *non-dominated* solutions has to be approximated by the heuristic search procedure.

We may use the dominance degrees of eq. 8 to rank the set M of multivariate data (vectors) given by the fitness values of a multiobjective optimization problem. Each element of M is assigned the maximum degree of being dominated by any other element of M, and the elements of M are sorted according to the ranking values in increasing order:

$$r_M(\vec{a}) = \max_{\vec{b} \in M \setminus \{\vec{a}\}} \mu_p(\vec{a},\vec{b}) \tag{9}$$

Note again that this definiton is related to a set. A *ranking value* of $\vec{a}$ within M can only be assigned with reference to a set M containing $\vec{a}$.

By sorting the elements of M according to the ranking values in increasing order (FPD ranking, FPD for Fuzzy-Pareto-Dominance), we obtain a partial ranking of the elements of M.

From the definition of the ranking scheme, it can be seen that an individual has two ways to reduce its comparison values: by increasing the objectives (thus increasing the denominator in the comparison values), or/and by being larger in some components than other vectors, i.e. being diverse from other vectors. Thus, both goals of evolutionary multi-objective optimization are met: to approach the Pareto front, and to maintain a diverse population.

The foregoing discussion leads to the (Fuzzy-Dominance-Driven) FDD algorithm, a Genetic Algorithm (GA) variant that employs the fuzzy ranking values of the fitness values (represented as vectors in case of multiobjective optimization) for defining selection operators. The algorithm and its components can be seen in fig. 5.

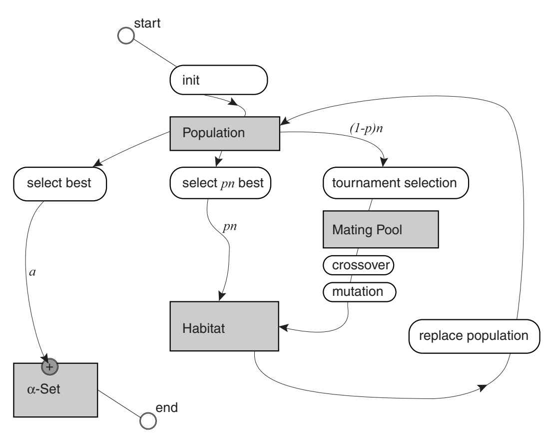

Fig. 5. Schematic view of FDD algorithm.

FDD maintains four pools of individuals:

- Population: contains n individuals as in standard GA.

- Mating Pool: Contains individual pairs that were selected for crossover operation.
- Habitat: This pool is composed of individuals from other pools and used to replace the population of generation n by generation $n+1$.
- α-Set: In this pool, all non-dominated individuals are collected. This pool also gives the output of the FDD algorithm.

After random initialization of the population, the FDD algorithm iteratively repeats the following steps until a stopping criteria (number of generations, size of α-Set) is met:

1) Rank population by FPD ordering of fitness vectors of the individuals in the population (see section 1).
2) Select *best individual* a from the ranked population (one individual with lowest ranking value) and conditionally add it to the α-set. Adding a to the α-set is only possible, when fitness of a is not dominated by the fitness of any individual already in the α-set, and if fitness of a is not equal to any individual's fitness there. In case a is added, all individuals in the α-set with fitness values dominated by fitness of a are removed from the α-set.
3) Add best pn of population individuals, according to FPD ordering ranking values, to the habitat ($0 \leq p \leq 1$).
4) Select $(1-p)n$ pairs from population by tournament selection, using ranking values of the ranked population for tournament decision (lower ranking value counts better), and put these pairs into mating pool.
5) Apply crossover and mutation to the individuals of the mating pool, and add these newly created individuals to the habitat as well.
6) Replace population by habitat.

The FDD algorithm acquires non-dominated (with respect to their fitness values) individuals in the α-set. In an evolutionary sense, those "FDD Pareto Set" approaches the Pareto front of the multiobjective optimization problem under study.

The algorithm has been verified by using test function MOP6 from the set suggested by Coello Coello [4].

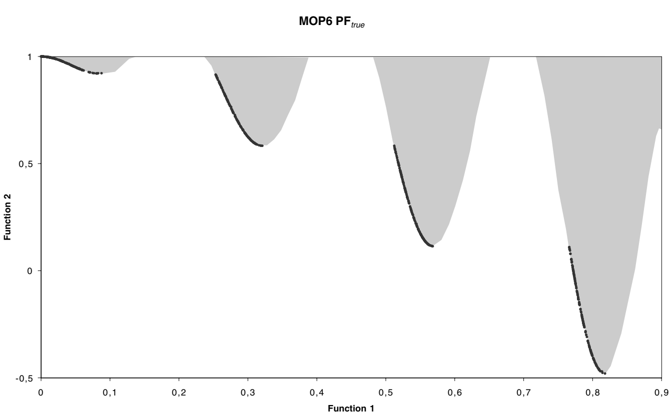

Fig. 6. Location of individuals in α-set after 2000 generations of FDD algorithm for MOP6 problem.

The FDD was tested against random search algorithm. The performance was measured as follows:

1) Let FDD run for k generations and take the fitness values of the α-set as set M_1.
2) Select $k \times n$ random domain values (with n the number of individuals of FDD algorithm) and compute all corresponding fitness values, giving set RM_2.
3) Compute the Pareto set M_2 of RM_2.
4) Compute the set M_3 of elements of M_2 that are not dominated by any element of M_1.

The relation of $|M_1|$ to $|M_3|$ gives a measure how FDD performs against random search.

MOP6([4], p. 111) is defined as follows:

$$F = (f_1(x,y), f_2(x,y))$$

where

$$\begin{aligned} f_1(x,y) &= x \\ f_2(x,y) &= (1+10y) * \\ &* \left[1 - \left[\frac{x}{1+10y}\right]^{\alpha} - \frac{x}{1+10y}\sin(2\pi q x)\right] \end{aligned} \tag{10}$$

with $0 \leq x, y \leq 1$ and the paramter choices $q = 4$ and $\alpha = 2$.

FDD was applied to this problem, with the following configuration:

- x and y values were encoded into bitstrings of size 40, with 20 bits for binary representation of each number.
- Population size was 50, with keeping 20 ($p = 0.4$) best from former generation in each new generation. The 20 best were selected by FPD ranking.
- Mating pool was obtained by tournament selection of 60 individuals according to FPD ranking. Two-point crossover was used, as well as bitwise one-point mutation with probability of 0.01.

TABLE II

PERFORMANCE OF FDD AGAINST RANDOM SEARCH. M_1 IS THE SET OF NON-DOMINATED FDD INDIVIDUALS AFTER n GENERATIONS, M_2 THE SET OF NON-DOMINATED INDIVIDUALS FOUND BY RANDOM SEARCH, AND M_3 THE SUBSET OF M_2 THAT IS NOT DOMINATED BY ANY INDIVIDUAL OF M_1. LISTED ARE AVERAGE VALUES OF SET SIZES AFTER 10 FDD RUNS. AFTER ABOUT 100 GENERATIONS, FDD OUTPERFORMS RANDOM SEARCH.

Generations	$\|M_1\|$	$\|M_2\|$	$\|M_3\|$
20	5.0	11.5	10.6
50	9.7	15.3	11.2
100	32.2	22.4	7.2
200	69.4	30.8	5.2
1000	411.2	122.0	0.0

Figure 6 gives the α-set fitness values of a FDD run after 2000 generations. The gray areas underlying the plot gives the range of MOP6 function values and were computed by Monte Carlo method with 5×10^7 test points. Note that fig. 6 only shows a part of the complete range of MOP6, containing the Pareto front. The α-set clearly has approached the Pareto front of the test problem. To illustrate the implicite niching of the FDD algorithm, fig. 7 shows the Pareto fronts of MOP6 problem, as approximated by the FDD algorithm and with higher values of q (where the value of q equals the number of different connected components of the Pareto front).

Table II shows the decrease of size of set $|M_3|$ (the randomly found individuals that are not dominated by any individual in the α-set) towards 0.

V. Conclusion

A generic fuzzy scheme for the ranking of multivariate data has been presented. The basic approach was to provide a fuzzification of the Pareto dominance relation. The approach to extend the logical relation itself was shown to be hard to handle and gives raise to a new class of functions, behaving like T-norms in some arguments, and like S-norms in other arguments. Instead of this, the scheme considered in this paper is based on a comparison function of two numbers. The comparison function values are fused by a T-norm for all components of two vectors, giving the comparison values. For each vector in a set of vectors, the smallest value of its comparison values with all other elements of the set is assigned to this vector as its ranking value. Then, all further processing is based on the ranking values alone. As a suitable comparison function, bounded division is identified. The application of the scheme to define a color morphology and an evolutionary multiobjective optimization algorithm has been demonstrated and shortly discussed. However, other applications are possible. Current studies consider the usage of the fuzzy ranking scheme in feature-based database retrieval. Here, the schema can be evoked to select probes from the most similar datasets.

Acknowledgment

The research presented in this paper has been supported by research funding from German BMBF, project "VisionIC."

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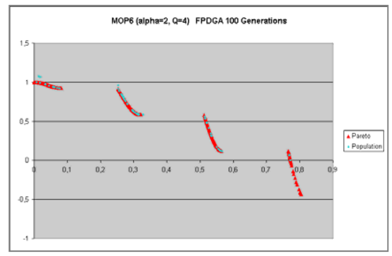
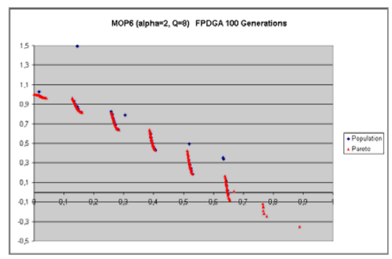
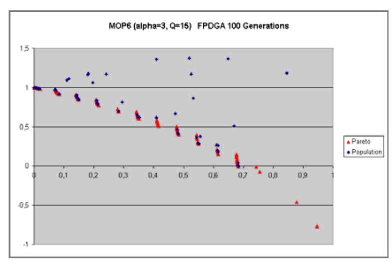


(a)



(b)



(c)

Fig. 7. Approximated Pareto fronts by FDD algorithm for MOP6 problem with higher values of q, i.e. higher diversity of Pareto front.